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TESIS DOCTORAL:

Local uniformization of codimension one foliations. Rational archimedean valuations

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Introducción

El objetivo de la presente memoria es demostrar el siguiente resultado

Teorema I. *Sea k un cuerpo de característica* 0 *y K/k una extensión de cuerpos finitamente generada. Sea $\mathcal{F}$ una foliación racional de codimensión uno de K/k. Dada una valoración k-racional arquimediana ν de K/k, existe un modelo proyectivo M de K/k tal que $\mathcal{F}$ es log-final en el centro de ν en M.*

La prueba de este teorema sigue las ideas clásicas de la uniformización local de Zariski, procediendo por inducción en el número de variables. La principal dificultad cuando tratamos con 1-formas es que la propiedad de integrabilidad se pierde durante el proceso de inducción. Para solucionarlo hemos estructurado nuestros resultados en términos de truncaciones (siguiendo la valoración) de funciones formales y formas diferenciales. Una parte importante de nuestro trabajo consiste en el control de una condición de integrabilidad parcial durante el proceso de truncación.

El Teorema I es un primer paso en la prueba de la siguiente conjetura, cuya demostración completa es el objeto de futuros trabajos:

Conjetura. *Sea k un cuerpo de característica* 0 *y K/k una extensión de cuerpos finitamente generada. Sea $\mathcal{F}$ una foliación racional de codimensión uno de K/k. Dada una valoración ν de K/k, existe un modelo proyectivo M de K/k tal que $\mathcal{F}$ es log-final en el centro de ν en M.*

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Consideremos una extensión de cuerpos finitamente generada K/k de grado de transcendencia $\operatorname{tr.deg}(K/k) = n$ sobre un cuerpo k de característica 0. Sea $\{z_1, z_2, \ldots, z_n\} \subset K$ una base de trascendencia. Tenemos la torre de cuerpos

$$k \subset k(z_1, z_2, \ldots, z_n) \subset K \ ,$$

donde K es una extensión algebraica finitamente generada (y separable ya que $\operatorname{char}(k) = 0$) de $k(z_1, z_2, \ldots, z_n)$. El módulo de diferenciales de Kähler $\Omega_{K/k}$ es un K-espacio vectorial de dimensión $n = \operatorname{tr.deg}(K/k)$ y $\{dz_1, dz_2, \ldots, dz_n\}$ conforma una base de dicho espacio. Una *foliación singular racional $\mathcal{F}$ de K/k* es un K-subespacio vectorial de dimensión uno de $\Omega_{K/k}$ tal que para toda 1-forma $\omega \in \mathcal{F}$ se satisface la condición de integrabilidad

$$\omega \wedge d\omega = 0 \ .$$

Esta definición de foliación coincide con la definición clásica de la geometría proyectiva compleja. Consideremos el espacio proyectivo complejo $\mathbb{P}^n_{\mathbb{C}}$ y una descomposición en cartas afines $\mathbb{P}^n_{\mathbb{C}} = U_0 \cup U_1 \cup \cdots \cup U_n$. Una foliación de codimensión uno de $\mathbb{P}^n_{\mathbb{C}}$ está dada por $n+1$ formas polinomiales homogéneas integrables

$$W_i = \sum_{j=1}^{n} P^i_j(z^i_1, z^i_2, \ldots, z^i_n)\, dz^i_j \ , \quad i = 0, 1, \ldots, n \ ,$$

definidas en las cartas afines $U_i \simeq \mathbb{C}[z^i_1, z^i_2, \ldots, z^i_n]$, de forma que

$$W_{i|_{U_i \cap U_j}} = G_{ij}\, W_{\ell|_{U_i \cap U_j}} \ ,$$

donde G_{ij} es una función racional invertible en $U_i \cap U_j$. El cuerpo de funciones de cada carta afín, y del propio $\mathbb{P}^n_{\mathbb{C}}$, es

$$K \simeq \mathbb{C}(z^i_1, z^i_2, \ldots, z^i_n)$$

para cualquier índice i. Todas las 1-formas W_i se pueden considerar como elementos de $\Omega_{K/\mathbb{C}}$. Cualquiera de ellas genera el mismo subespacio vectorial de dimensión 1

$$\mathcal{F} =< W_i >\subset \Omega_{K/\mathbb{C}} \ ,$$

que es una foliación racional de codimensión uno de $K/\mathbb{C}$ según nuestra definición.

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Un modelo proyectivo de K/k es una k-variedad proyectiva M, en el sentido de la teoría de esquemas, de forma que $K = \kappa(M)$ es su cuerpo de funciones racionales. Tomemos un punto $Y \in M$ regular k-racional, es decir, un punto tal que el anillo local $\mathcal{O}_{M,Y}$ es regular y su cuerpo residual es $\kappa_{M,Y} \simeq k$. Un sistema de generadores $z_1, z_2, \ldots, z_n$ del ideal maximal $\mathfrak{m}_{M,Y}$ es a su vez una base de transcendencia de K/k, luego proporciona una base $dz_1, dz_2, \ldots, dz_n$ de $\Omega_{K/k}$. Consideremos un sistema de generadores de $\mathfrak{m}_{M,Y}$ de la forma $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y})$, donde $\boldsymbol{x} = (x_1, x_2, \cdots, x_r)$ e $\boldsymbol{y} = (y_1, y_2, \cdots, y_{n-r})$. Sea $\Omega_{\mathcal{O}_{M,Y}/k}(\log \boldsymbol{x})$ el $\mathcal{O}_{M,Y}$-submódulo de $\Omega_{K/k}$ generado por $\Omega_{\mathcal{O}_{M,Y}/k}$ y las diferenciales logarítmicas

$$\frac{dx_1}{x_1}, \frac{dx_2}{x_2}, \ldots, \frac{dx_r}{x_r} \ .$$

Tenemos que $\Omega_{\mathcal{O}_{M,Y}/k}(\log \boldsymbol{x})$ es un $\mathcal{O}_{M,Y}$-módulo libre de rango n generado por

$$\frac{dx_1}{x_1}, \frac{dx_2}{x_2}, \ldots, \frac{dx_r}{x_r}, dy_1, dy_2, \ldots, dy_{n-r} \ .$$

Sea $\mathcal{F}$ una foliación singular racional de K/k. Definimos

$$\mathcal{F}_{M,Y}(\log \boldsymbol{x}) = \mathcal{F} \cap \Omega_{\mathcal{O}_{M,Y}/k}(\log \boldsymbol{x}) \ .$$

$\mathcal{F}_{M,Y}(\log \boldsymbol{x})$ es un $\mathcal{O}_{M,Y}$-módulo libre de rango 1 generado por una 1-forma integrable

$$\omega = \sum_{i=1}^{r} a_i \frac{dx_i}{x_i} + \sum_{j=1}^{n-r} b_j dy_j \ ,$$

donde los coeficientes $a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_{n-r} \in \mathcal{O}_{M,Y}$ no tienen factor común. Decimos que

1. $\mathcal{F}$ es $\boldsymbol{x}$*-log elemental* en $Y \in M$ si $(a_1, a_2, \ldots, a_r) = \mathcal{O}_{M,Y}$;
2. $\mathcal{F}$ is $\boldsymbol{x}$*-log canónica* en $Y \in M$ si $(a_1, a_2, \ldots, a_r) \subset \mathfrak{m}_{M,Y}$ y además

$$(a_1, a_2, \ldots, a_r) \not\subset (x_1, x_2, \ldots, x_r) + \mathfrak{m}_{M,Y}^2 \ .$$

Decimos que $\mathcal{F}$ es *x-log final* en $Y \in M$ si es $\boldsymbol{x}$-log elemental o $\boldsymbol{x}$-log canónica. Finalmente, diremos que $\mathcal{F}$ es *log-final* en $Y \in M$ si es $\boldsymbol{x}$-log final para algún sistema de generadores $(\boldsymbol{x}, \boldsymbol{y})$ de $\mathfrak{m}_{M,Y}$.

La propiedad de ser log-final es la versión algebraica del concepto de singularidad pre-simple del caso analítico complejo ([21],[7],[5]). Recordemos brevemente esta definición. Consideremos una foliación de $(\mathbb{C}^2, \mathbf{0})$ dada localmente por

$$a(x,y)dx + b(x,y)dy = 0 \ .$$

El origen $(0,0)$ es una singularidad pre-simple si la foliación es no singular (una de las series $a(x,y)$ o $b(x,y)$ es una unidad) o si la matriz jacobiana

$$\begin{pmatrix} \partial b/\partial x(0,0) & -\partial a/\partial x(0,0) \\ \partial b/\partial y(0,0) & -\partial a/\partial y(0,0) \end{pmatrix}$$

es no nilpotente. En esta situación siempre podemos tomar coordenadas analíticas x', y' tales que la foliación esté dada localmente por

$$a'(x',y')\frac{dx'}{x'} + b'(x',y')dy' = 0 \ ,$$

donde $a'(x',y') = y' + \cdots$, luego la foliación es x'-log-final en el origen, con respecto a las coordenadas analíticas (x',y'). Un análisis detallado puede verse en [8]. En general, para foliaciones de espacios ambiente complejos de dimensión arbitraria, el concepto de singularidad pre-simple introducido en [7] y [5] es equivalente a la propiedad de ser log-final.

En el caso de foliaciones sobre variedades algebraicas de dimensión 2 o 3, el teorema demostrado en esta memoria, así como la propia conjetura, son consecuencia de los siguientes resultados globales de reducción de singularidades (ver [21] para el caso en dimensión 2 y [5] para la dimensión 3):

Teorema (A. Seidenberg, 1968; F. Cano 2004)**.** *Sea $\mathcal{F}$ una foliación singular de codimensión uno de $(\mathbb{C}^n, 0)$, $n = 2, 3$. Existe una composición finita de blow-ups*

$$(\mathbb{C}^n, 0) \leftarrow (M_1, Z_1) \leftarrow \cdots \leftarrow (M_N, Z_N) = (M, Z)$$

tal que $\mathcal{F}$ es log-final en todo punto $Y \in Z$.

En el caso de espacios ambiente de dimensión $n \geq 4$ la reducción global de singularidades de foliaciones de codimensión uno es un problema abierto.

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La resolución de singularidades de variedades algebraicas sobre un cuerpo base de característica 0 fue probada por Hironaka [14].

Teorema (Reducción de singularidades de Hironaka, 1964)**.** *Sea K/k una extensión de cuerpos finitamente generada, donde k tiene característica* 0*. Existe un modelo proyectivo no singular M de K/k.*

Previamente al trabajo de Hironaka, el problema había sido resuelto en dimensión menor o igual que 3. El caso de curvas complejas ya fue tratado por Newton en 1676. Para superficies, la primera prueba rigurosa se debe a Walker en 1935 [25]. El caso de 3-variedades fue resuelto por Zariski en 1944 [27]. Antes de este resultado, Zariski había probado la uniformización local para variedades algebraicas en característica cero en dimensión arbitraria [26].

Teorema (Uniformización local de Zariski, 1940). *Sea K/k una extensión de cuerpos finitamente generada, donde k tiene característica 0, y sea ν una valoración de K/k. Existe un modelo proyectivo no singular M de K/k tal que el centro de ν en M es un punto regular.*

En [28], Zariski demuestra la compacidad de la superficie de Riemann-Zariski (el espacio de todas las valoraciones de K/k), lo cual implica que un número finito de modelos proyectivos son suficientes para obtener uniformización local para cualquier valoración. Tras ello, logra el resultado global pegando modelos proyectivos, método que solo funciona en dimensión menor o igual que tres.

El problema general de resolución de singularidades tiene una larga historia tras los trabajos de Zariski e Hironaka. El caso de espacios analíticos complejos fue tratado por Aroca, Hironaka y Vicente [3]. La larga y compleja demostración de Hironaka ha sido analizada cuidadosamente, con énfasis en la constructividad y las propiedades funtoriales por Villamayor [24], Bierstone y Milman [4] y otros.

Una de las claves en la resolución de singularidades es la teoría del contacto maximal y su versión diferencial, desarrollada por Giraud [13]. Éste es uno de los puntos de partida del resultado conocido más fuerte en característica positiva, debido a Cossart y Piltant, quienes probaron la resolución de singularidades de 3-variedades en característica positiva [10, 11]. Éste trabajo mejora los resultados de Abhyankar, quien probó la resolución para superficies [1], y para 3-variedades en el caso de cuerpos de característica mayor o igual que 7 [2]. Todos estos resultados en característica positiva pasan por la uniformización local.

Otro problema relacionado es la monomialización de morfismos en característica cero, tratado por Cutkosky [12]. Las dificultades en este caso son cercanas a las que aparecen en el tratamiento de campos de vetores.

La reducción de singularidades de campos de vectores en dimensión dos fue obtenida por Seidenberg [21]. En dimensión 3, hay resultados parciales debidos a Cano [6], y más tarde este autor junto a Roche y Spivakovsky prueba la reducción global vía uniformización local en [9], utilizando la versión axiomática del pegado de Zariski desarrollada por Piltant [18]. Recientemente McQuillan y Panazzolo tratan el caso de dimensión 3 desde un punto de vista no birracional [17].

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En esta memoria tratamos el caso de k-valoraciones racionales de rango 1. En el tratamiento clásico de Zariski del problema de Uniformización Local, éste es el punto de partida, y en él se concentran las principales dificultades algorítmicas y combinatorias. Esperamos que la situación sea similar en el caso de foliaciones de codimensión uno, y que partiendo del resultado obtenido en esta tesis podamos completar la prueba de la conjetura general en futuros trabajos.

Una valoración $\nu : K^* \to \Gamma$ de K/k se dice k-racional si su cuerpo residual κ_ν es isomorfo al cuerpo base k. El rango $\mathrm{rank}(\nu)$ es 1 si y sólo si existe una inclusión de grupos ordenados $\Gamma \subset (\mathbb{R}, +)$.

Sea M un modelo proyectivo de K/k. El centro de ν en M es el único punto $Y \in M$ tal que para cualquier $\phi \in \mathcal{O}_{M,Y}$ se tiene

$$\nu(\phi) \geq 0 \quad \text{y además} \quad \nu(\phi) > 0 \Leftrightarrow \phi \in \mathfrak{m}_{M,Y} \ .$$

Dicho punto siempre existe y es único (ver [26] o [19]). Además, se tiene una torre de cuerpos

$$k \subset \kappa_{M,Y} \subset \kappa_\nu \ .$$

Debido a que en nuestro trabajo solo consideramos valoraciones k-racionales tenemos que $k = \kappa_{M,Y}$ y por lo tanto los centros de ν en cada modelo proyectivo son puntos k-racionales (y en particular cerrados).

El *rango racional* $\mathrm{rat.rk}(\nu)$ es la dimensión sobre $\mathbb{Q}$ de $\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$. La desigualdad de Abhyankar garantiza que $\mathrm{rat.rk}(\nu) \leq \mathrm{tr.deg}(K/k)$. El rango racional se corresponde con el máximo número de elementos $\phi_1, \phi_2, \ldots, \phi_r \in K^*$ con valores $\mathbb{Z}$-independientes $\nu(\phi_1), \nu(\phi_2), \ldots, \nu(\phi_r) \in \Gamma$.

Todos los resultados técnicos que utilizamos están enunciados en términos de *modelos locales regulares parametrizados.* Un modelo local regular parametrizado $\mathcal{A}$ de $K/k, \nu$ es un par

$$\mathcal{A} = (\mathcal{O}, (\boldsymbol{x}, \boldsymbol{y}))$$

tal que

1. existe un modelo proyectivo M de K/k tal que el centro Y de ν es un punto regular de M y además $\mathcal{O} = \mathcal{O}_{M,Y}$;

2. la lista $(x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_{n-r})$, donde $r = \mathrm{rat.rk}(\nu)$, es un sistema regular de parámetros de $\mathcal{O}$ y además los valores $\nu(x_1), \nu(x_2), \ldots, \nu(x_r)$ son $\mathbb{Z}$-independientes.

La existencia de modelos locales regulares parametrizados se prueba haciendo uso del resultado de resolución global de singularidades de Hironaka [14]. Dicha prueba se encuentra detallada en [9], trabajo en el que se introducen dichos modelos.

Acorde con esta terminología, dada $\mathcal{F}$ foliación racional de codimensión uno de K/k, denotamos por $\mathcal{F}_{\mathcal{A}}$ a

$$\mathcal{F}_{\mathcal{A}} = \mathcal{F} \cap \Omega_{\mathcal{O}/k}(\log \boldsymbol{x}) = \mathcal{F}_{M,Y}(\log \boldsymbol{x}) \ .$$

Diremos que $\mathcal{F}$ es $\mathcal{A}$-*final* si $\mathcal{F}_{\mathcal{A}}$ es $\boldsymbol{x}$-log final.

Usaremos transformaciones entre modelos locales regulares parametrizados $\mathcal{A} \to \mathcal{A}'$, llamadas *operaciones básicas*, las cuales tienen un morfismo subyacente $\mathcal{O} \to \mathcal{O}'$ que puede ser o bien un blow-up o bien el morfismo identidad. Las describimos a continuación:

- *Cambios de coordenadas ordenados.* El morfismo subyacente $\mathcal{O} \to \mathcal{O}'$ es la identidad. Fijado un índice $0 \leq \ell \leq n - r$ se define una nueva coordenada

$$y'_\ell = y_\ell + \psi(\boldsymbol{x}, y_1, y_2, \ldots, y_{\ell-1}),$$

donde $\psi(\boldsymbol{x}, y_1, y_2, \ldots, y_{\ell-1}) \in k[\boldsymbol{x}, y_1, y_2, \ldots, y_{\ell-1}]$ se expresa de forma

$$\psi(\boldsymbol{x}, y_1, y_2, \ldots, y_{\ell-1}) = \sum_I \boldsymbol{x}^I \psi_I(y_1, y_2, \ldots, y_{\ell-1})$$

con $\nu(\boldsymbol{x}^I) \geq \nu(y_\ell)$ si $\psi_I \neq 0$.

- *Blow-ups con centros de codimensión dos.* El centro del blow-up será o bien $x_i = x_j = 0$ o bien $x_i = y_j = 0$. El anillo $\mathcal{O}'$ está determinado por

$$\mathcal{O}' = \mathcal{O}[\boldsymbol{x}', \boldsymbol{y}']_{(\boldsymbol{x}', \boldsymbol{y}')}$$

donde las coordenadas $(\boldsymbol{x}', \boldsymbol{y}')$ se obtienen de la siguiente forma:

1. Si el centro es $x_i = x_j = 0$ y además $\nu(x_i) < \nu(x_j)$, entonces tomamos $x'_j := x_j/x_i$.
2. Si el centro es $x_i = y_j = 0$ y además $\nu(x_i) < \nu(y_j)$, entonces tomamos $y'_j := y_j/x_i$.
3. Si el centro es $x_i = y_j = 0$ y además $\nu(x_i) > \nu(y_j)$, entonces tomamos $x'_i := x_i/y_j$.
4. Si el centro es $x_i = y_j = 0$ y además $\nu(x_i) = \nu(y_j)$, entonces tomamos $y'_j := y_j/x_i - \xi$, donde $\xi \in k^*$ es la única constante tal que $\nu(y_j/x_i - \xi) > 0$.

El último caso se trata de un *blow-up con translación.* El resto de casos son *blow-up combinatorios.*

El Teorema I es consecuencia del siguiente resultado establecido en términos de modelos locales regulares parametrizados:

Teorema II. *Sea k un cuerpo de característica 0 y K/k una extensión de cuerpos finitamente generada. Sea $\mathcal{F}$ una foliación racional de codimensión uno de K/k. Dada una valoración k-racional arquimediana ν de K/k y un modelo local regular parametrizado $\mathcal{A}$ de $K/k, \nu$, existe una sucesión finita de transformaciones básicas*

$$\mathcal{A} = \mathcal{A}_0 \to \mathcal{A}_1 \to \cdots \to \mathcal{A}_N = \mathcal{B}$$

tal que $\mathcal{F}$ es $\mathcal{B}$-final.

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Consideraremos sistemáticamente el completado formal $\hat{\mathcal{O}}$ de el anillo local $\mathcal{O}$. Un primer motivo para ello es de naturaleza práctica, ya que al ser

$$\hat{\mathcal{O}} \simeq k[[\boldsymbol{x}, \boldsymbol{y}]] ,$$

podemos considerar los elementos de $\mathcal{O}$ como series formales. Un segundo motivo para considerar el completado formal se debe al hecho de que las soluciones de ecuaciones diferenciales con coeficientes en $\mathcal{O}$ no se encuentran necesariamente en el propio anillo $\mathcal{O}$ (esto sucede incluso si trabajamos en la categoría analítica).

Ilustremos este hecho con un ejemplo. Si nuestro "objeto problema" es una función $f \in \mathcal{O}$, tras una secuencia finita de transformaciones básicas obtenemos un modelo $\mathcal{A}' = (\mathcal{O}', (\boldsymbol{x}', \boldsymbol{y}'))$ tal que

$$f = \boldsymbol{x}'^{\boldsymbol{p}} U \ , \quad U \in \mathcal{O}' \setminus \mathfrak{m}' \ .$$

Este hecho es consecuencia directa de la Uniformización Local de Zariski. Por completitud, incluiremos la demostración, que a su vez nos servirá como guía para el tratamiento de las 1-formas diferenciales. Si consideramos la foliación dada por $df = 0$ tenemos que es $\mathcal{A}'$-final, y en particular es $\boldsymbol{x}$-log elemental. Ésta propiedad también se cumple para foliaciones que tienen integral primera: siempre puede alcanzarse un modelo en el cual sea $\boldsymbol{x}$-log elemental. Sin embargo, esto no ocurre en general. Ya en dimensión dos encontramos un ejemplo con la Ecuación de Euler:

$$(y - x)\frac{dx}{x} - x dy = 0 \ .$$

La foliación de $(\mathbb{C}^2, \mathbf{0})$ dada por esta ecuación es x-log canónica. Además, tiene una curva formal invariante de ecuación $\hat{y} = 0$ donde

$$\hat{y} = y - \sum_{n=0}^{\infty} n!\, x^{n+1} \ .$$

Se tiene la igualdad

$$(y - x)\frac{dx}{x} - x dy = \hat{y} \left(\frac{dx}{x} - x \frac{d\hat{y}}{\hat{y}} \right) \ ,$$

luego si admitimos el uso de coordenadas formales $(x, \hat{y})$, la foliación estaría dada por

$$\frac{dx}{x} - x \frac{d\hat{y}}{\hat{y}} = 0 \ ,$$

y por lo tanto dicha foliación sería "$x\hat{y}$-log elemental". Si consideramos la valoración de $\mathbb{C}(x, y)/\mathbb{C}$ dada por

$$\nu(f(x, y)) = \mathrm{ord}_t(f(t, \sum_{n=0}^{\infty} n! t^{n+1}))$$

podemos comprobar que no es posible obtener por medio de transformaciones básicas un modelo de forma que la foliación sea x-log elemental. Independientemente de las transformaciones básicas que hagamos la foliación continuará siendo x-log canónica. De hecho, ν es la única valoración de $\mathbb{C}(x, y)/\mathbb{C}$ que tiene esta propiedad. Esto se debe al hecho de que ν sigue los puntos infinitamente próximos de la curva formal invariante $\hat{y} = 0$.

En cierto sentido podríamos pensar que la forma diferencial $\omega = x dy - (y - x)dx/x$ tiene "valor infinito" en relación a ν. La propiedad de "tener valor infinito" puede darse también para funciones formales $\hat{f} \in \hat{\mathcal{O}} \setminus \mathcal{O}$ (en este ejemplo $\hat{y} \in \hat{\mathcal{O}}$ tiene "valor infinito"), pero nunca puede ocurrir para elementos $f \in \mathcal{O} \setminus \{0\}$. Nótese que aunque ω tiene "valor infinito" sus coeficientes no.

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Una de las principales diferencias entre nuestro procedimiento y el tratamiento clásico de Zariski es la consideración sistemática de truncaciones relativas a un elemento del grupo de valores. Este método es esencial para nosotros, pues gracias a él podemos controlar la condición de integrabilidad dentro del proceso de inducción. Además, este método es aplicable a objetos formales en general, tanto funciones formales como 1-formas diferenciales con coeficientes formales.

Sea $\mathcal{A} = (\mathcal{O}, (\boldsymbol{x}, \boldsymbol{y}))$ un modelo local regular parametrizado para K, ν, donde $\boldsymbol{x} = (x_1, x_2, \ldots, x_r)$ e $\boldsymbol{y} = (y_1, y_2, \ldots, y_{n-r})$. Para cada índice $0 \leq \ell \leq n-r$, consideremos el anillo de series

$$R_{\mathcal{A}}^{\ell} = k[[\boldsymbol{x}, y_1, y_2, \ldots, y_\ell]] \ .$$

Nótese que $R_{\mathcal{A}}^{n-r} \simeq \hat{\mathcal{O}}$. A su vez, para cada índice $0 \leq \ell \leq n-r$ definimos el $R_{\mathcal{A}}^{\ell}$-módulo

$$N_{\mathcal{A}}^{\ell} = \bigoplus_{i=1}^{r} R_{\mathcal{A}}^{\ell} \frac{dx_i}{x_i} \ \oplus \ \bigoplus_{j=1}^{\ell} R_{\mathcal{A}}^{\ell} dy_j \ .$$

Cualquier elemento $\omega \in N_{\mathcal{A}}^{\ell}$ puede ser escrito de manera única como $\omega = \sum_I \boldsymbol{x}^I \omega_I$, donde

$$\omega_I = \sum_{i=1}^{r} a_{I,i}(y_1, y_2, \ldots, y_\ell) \frac{dx_i}{x_i} + \sum_{j=1}^{\ell} b_{I,j}(y_1, y_2, \ldots, y_\ell) dy_j \ .$$

Definimos el *valor explícito* $\nu_{\mathcal{A}}(\omega)$ como el mínimo entre los valores $\nu(\boldsymbol{x}^I)$ tales que $\omega_I \neq 0$.

Fijemos un elemento del grupo de valores $\gamma \in \Gamma$. Tomemos $\omega \in N_{\mathcal{A}}^{\ell}$ y sea x^{I_0} el monomio que satisface $\nu_{\mathcal{A}}(\omega) = \nu(x^{I_0})$. Decimos que ω es *γ-final in $\mathcal{A}$* si se da una de las siguientes situaciones:

- *Caso γ-final dominante:* $\nu_{\mathcal{A}}(\omega) \leq \gamma$ y al menos uno de los coeficientes $a_{I_0,i}$ satisface $a_{I_0,i}(0) \neq 0$ (dicho de otro modo, si ω_{I_0} es $\boldsymbol{x}$-log elemental).
- *Caso γ-final recesivo:* $\nu_{\mathcal{A}}(\omega) = \nu(x^{I_0}) > \gamma$.

La prueba del Teorema II (y por lo tanto del Teorema I) se deriva del siguiente resultado enunciado en términos de truncaciones:

Teorema III (Uniformización Local Truncada)**.** *Sea $\omega \in N_{\mathcal{A}}^{\ell}$ y fijemos $\gamma \in \Gamma$. Si se satisface*

$$\nu_{\mathcal{A}}(\omega \wedge d\omega) \geq 2\gamma \ ,$$

entonces existe una composición finita de transformaciones básicas $\mathcal{A} \to \mathcal{B}$, que no afecta a las variables y_j con $j > \ell$, de forma que ω es γ-final en $\mathcal{B}$.

La parte principal y de mayor dificultad técnica de este trabajo, Capítulos 6 y 7, está dedicada a probar el Teorema III. En el Capítulo 8 se muestra como concluir el Teorema I como consecuencia del Teorema III.

- - -

Probaremos el Teorema III por inducción en el número de variables dependientes ℓ. En lugar de usar composiciones arbitrarias de transformaciones básicas, nos restringiremos a las *transformaciones ℓ-anidadas*, las cuales definimos por inducción. Una *transformación* 0*-anidada* es una composición finita de blow-ups de tipo $x_i = x_j = 0$ (en particular todos ellos son combinatorios). Una *transformación ℓ-anidadas* es una composición finita de transformaciones $(\ell-1)$-anidadas, *ℓ-paquetes de Puiseux* y cambios ordenados de la ℓ-ésima variable. Este tipo de transformaciones están estrechamente relacionadas con las *monoidal transform sequences* y las *uniformizing transform sequences* usadas por Cutkosky en [12].

Dada una variable dependiente y_ℓ, su valor depende linealmente del valor de las variables independientes $\boldsymbol{x}$. Esto quiere decir que existen enteros $d \geq 1$ y $p_1, \ldots, p_r$ tales que

$$d\nu(y_\ell) = p_1\nu(x_1) + \cdots + p_r\nu(x_r) \ ,$$

o lo que es lo mismo, la *función racional de contacto* y_ℓ^d/x^p tiene valor nulo. Dado que la valoración es racional, existe una única constante $\xi \in k^*$ tal que $y_\ell^d/x^p - \xi$ tiene valor positivo. Un *ℓ-paquete de Puiseux* $\mathcal{A} \to \mathcal{A}'$ es cualquier composición finita de blow-ups de modelos locales regulares parametrizados con centros del tipo $x_i = x_j = 0$ o $x_i = y_\ell = 0$ tal que todos los blow-up sin combinatorios excepto el último. En particular, se tiene que $y'_\ell = y_\ell^d/\boldsymbol{x}^{\boldsymbol{p}} - \xi$. La existencia de los paquetes de Puiseux sigue de la resolución de singularidades del ideal binomial $(y_\ell^d - \xi\boldsymbol{x}^{\boldsymbol{p}})$.

Los paquetes de Puiseux fueron introducidos en [9] para el tratamiento de campos de vectores. En el caso de dimensión dos, los paquetes de Puiseux están directamente relacionados con los pares de Puiseux de las ramas analíticas que sigue la valoración. Otro concepto relacionado con los paquetes de Puiseux muy utilizado en teoría de valoraciones son los *polinomios clave* (ver [**?**] o [**?**, 18, **?**, **?**]).

Los enunciados $T_3(\ell)$, $T_4(\ell)$ y $T_5(\ell)$ establecidos más adelante se prueban por inducción en ℓ. El enunciado $T_3(\ell)$ trata sobre 1-formas con coeficientes formales. En particular, el Teorema III es equivalente a $T_3(n-r)$. El euncido $T_4(\ell)$ trata sobre funciones formales, mientras que $T_5(\ell)$ concierne a pares función-forma - de hecho, este resultado se puede enunciar, y probar, de forma más general, para listas finitas de funciones y formas sin dificultad añadida, pero es esta formulación precisa la que usaremos dentro de nuestro argumento de inducción - .

El concepto de valor explícito definido para 1-formas se extiende de forma directa al caso de funciones, pares forma-función y p-formas para $p \geq 2$. Del mismo modo, fijado un valor $\gamma \in \Gamma$ extendemos la definición de 1-forma γ-final al caso de funciones y de pares forma-función.

Sean $F \in R_{\mathcal{A}}^\ell$ una función formal y $\omega \in N_{\mathcal{A}}^\ell$ una 1-forma. Establecemos los siguientes enunciados:

$T_3(\ell)$: Supongamos que $\nu_{\mathcal{A}}(\omega\wedge d\omega) \geq 2\gamma$. Existe una transformación ℓ-anidada $\mathcal{A} \to \mathcal{B}$ tal que ω es γ-final en $\mathcal{B}$.

$T_4(\ell)$: Existe una transformación ℓ-anidada $\mathcal{A} \to \mathcal{B}$ tal que F es γ-final en $\mathcal{B}$.

$T_5(\ell)$: Supongamos que $\nu_{\mathcal{A}}(\omega\wedge d\omega) \geq 2\gamma$. Existe una transformación ℓ-anidada $\mathcal{A} \to \mathcal{B}$ tal que el par (ω, F) es γ-final en $\mathcal{B}$.

Los enunciados $T_3(0)$, $T_4(0)$ y $T_5(0)$ se derivan directamente de el control del Poliedro de Newton de un ideal por blow-ups combinatorios [23].

Como hipótesis de inducción asumiremos que los enunciados $T_3(k)$, $T_4(k)$ y $T_5(k)$ son ciertos para todo $k \leq \ell$. Veremos que $T_3(\ell+1)$ implica $T_4(\ell+1)$ y $T_5(\ell+1)$, aunque en el Capítulo 5 incluimos la demostración de $T_4(\ell+1)$ ya que nos servirá como guía para el caso de formas diferenciales. El paso más difícil será probar $T_3(\ell+1)$ haciendo uso de la hipótesis de inducción.

- - -

Asumiendo la hipótesis de inducción, dividiremos la prueba de $T_3(\ell+1)$ en dos etapas:

1. Proceso de *γ-preparación* de una 1-forma $\omega \in N_{\mathcal{A}}^{\ell+1}$ por medio de transformaciones ℓ-anidadas (Capítulo 6).

2. Obtención de formas γ-finales. Definiremos invariantes asociados a una 1-forma γ-preparada $\omega \in N_{\mathcal{A}}^{\ell+1}$ y mediante su control determinaremos una transformación $(\ell+1)$-anidada de modo que ω sea γ-final en el modelo local regular parametrizado obtenido (Capítulo 7).

A continuación damos una descripción breve de estas etapas.

- - -

A modo ilustrativo, mostraremos como obtener la Uniformización Local de funciones mediante el método de truncaciones. Las dificultades que se presentan en el proceso de γ-preparación de una 1-forma no aparecen en el caso de funciones. La diferencia principal radica en la naturaleza de los objetos a los cuales podemos aplicar la hipótesis de inducción. Dada una función $F \in R_{\mathcal{A}}^{\ell+1}$ podemos expresarla como una serie en la última variable dependiente

$$F = \sum_{s \geq 0} y_{\ell+1}^s F_s(\boldsymbol{x}, y_1, y_2, \ldots, y_\ell) \ .$$

Los coeficientes F_s pertenecen a $R_{\mathcal{A}}^{\ell}$ y por tanto podemos hacer uso de $T_4(\ell)$ en su tratamiento. Sin embargo, dada una 1-forma $\omega \in N_{\mathcal{A}}^{\ell+1}$, si la "descomponemos" del mismo modo

$$\omega = \sum_{s \geq 0} y_{\ell+1}^s \omega_s(\boldsymbol{x}, y_1, y_2, \ldots, y_\ell) \ ,$$

los "coeficientes" ω_s no pertenecen a $N_{\mathcal{A}}^{\ell}$, luego no podremos hacer uso de $T_3(\ell)$ directamente.

Ya que trabajamos por inducción en el parámetro ℓ denotaremos $z = y_{\ell+1}$ e $\boldsymbol{y} = (y_1, y_2, \ldots, y_\ell)$. Como anteriormente, dada $F \in R_{\mathcal{A}}^{\ell+1}$ consideramos su descomposición $F = \sum z^s F_s$. Usando esta descomposición definimos el *Polígono de Newton Truncado* $\mathcal{N}(F; \mathcal{A}; \gamma)$ como la envolvente positivamente convexa en $\mathbb{R}^2$ de los puntos $(0, \gamma/\nu(z))$, $(\gamma, 0)$ y los $(\nu_{\mathcal{A}}(F_s), s)$ para $s \geq 0$. Del mismo modo definimos el *Polígono de Newton Dominante Truncado* $\mathrm{Dom}\mathcal{N}$, esta vez considerando los puntos $(0, \gamma/\nu(z))$, $(\gamma, 0)$ y únicamente los $(\nu_{\mathcal{A}}(F_s), s)$ tales que F_s sea $(\gamma - s\nu(z))$-final dominante. La función F está γ-preparada en $\mathcal{A}$ si

$$\mathcal{N} = \mathrm{D}om\mathcal{N}$$

y además $(\nu_{\mathcal{A}}(F_s), s)$ es un punto interior de $\mathcal{N}$ para todo s tal que F_s no es dominante.

La γ-preparación de una función no presenta grandes dificultades. Primero, dado que una transformación ℓ-anidada no mezcla los niveles F_s entre ellos, sin hacer uso de la hipótesis de inducción, podemos aplicar una transformación ℓ-anidada de forma que obtengamos el mayor número posible de niveles $(\gamma - s\nu(z))$-finales. Tras ello, la hipótesis de inducción nos permite alcanzar una situación γ-preparada directamente: basta aplicar $T_4(\ell)$ a los niveles F_s no dominantes tales que $s \leq \gamma/\nu(z)$.

Una vez que tenemos que la función F está γ-preparada tomamos como invariante la *altura crítica* χ, la cual se corresponde con el punto más alto del lado de $\mathcal{N}$ con pendiente $-1/\nu(z)$ (ver Sección 5.2).

Ahora analizamos el comportamiento de χ tras aplicar un z-paquete de Puiseux, con función racional de contacto z^d/x^p, seguido de una nueva γ-preparación.

Si el *exponente de ramificación* d es estrictamente mayor que uno obtendremos $\chi' < \chi$. Si en algún momento llegamos a que la altura crítica sea 0, tras un z-paquete de Puiseux y una γ-preparación estaremos en una situación γ-final.

Por lo tanto, solo resta considerar el caso de tener $d = 1$ y $\chi > 0$ indefinidamente. En esta situación, cada z-paquete de Puiseux hace aumentar el valor explícito de F. Si en algún momento dicho valor sobrepasa γ habremos alcanzado una situación γ-final recesiva. Sin embargo, puede ocurrir que esto no suceda y el valor se "acumule" antes de alcanzar γ. Este fenómeno se debe a la posibilidad de tener $d \geq 2$ antes de la γ-preparación, pero $d = 1$ tras ella. En esta situación recurriremos a un cambio de coordenadas de Tschirnhausen parcial (un tipo concreto de cambio ordenado de variables). En el caso de formas diferenciales también habremos de recurrir a este tipo de cambios de coordenadas, solo que su determinación es más sutil.

Ilustremos esta situación con un ejemplo en dimensión tres. Definimos por recurrencia los siguientes elementos de $\mathbb{C}(x, y, z)$ para $j \geq 0$:

$$f_{j+1} = \frac{f_j}{g_j} \ , \quad g_{j+1} = \frac{g_j^2}{f_j} - 1 \ , \quad h_{j+1} = \frac{g_j h_j}{f_j} - 1 \ ,$$

donde

$$f_0 = x \ , \quad g_0 = y \ , \quad h_0 = z \ .$$

Sea ν una valoración de $\mathbb{C}(x, y, z)/\mathbb{C}$ tal que

$$\nu(x) = 1 \ , \quad \nu(y) = \nu(z) = \frac{1}{2} \ , \quad \nu(y - z) = 1$$

y además existe una serie trascendente $\phi = \sum_{k=1}^{\infty} c_k x^k$ tal que

$$\nu\left(y - z - \sum_{k=1}^{t} c_k x^k\right) = t + 1 \quad \text{para todo } t \geq 1 \ .$$

Tenemos que $\mathcal{A} = (\mathcal{O}, (x, y, z))$ donde $\mathcal{O} = \mathbb{C}[z, y, z]_{(x,y,z)}$ es un modelo local regular prametrizado de $\mathbb{C}(x, y, z)/\mathbb{C}, \nu$. Usando las relaciones de recurrencia se concluye que para cualquier $j \geq 0$ tenemos

$$\nu(f_j) = 2^{-j} \ , \quad \nu(g_j) = \nu(h_j) = 2^{-j-1} \quad \text{y} \quad \nu(g_j - h_j) = 2^{-j} \ .$$

Consideremos como objeto problema la función formal $F \in \mathbb{C}[[x, y, z]]$ dada por

$$F = z - y - \phi$$

y fijemos un valor $\gamma \geq 1$. La función F no está γ-preparada (nótese que el nivel $F_0 = -y - \phi$ tiene valor explicito $0 < \gamma$ y no es 0-final dominante). Para γ-preparar F basta con aplicar un y-paquete de Puiseux $\mathcal{A} \to \mathcal{A}'_1$, compuesto por dos blow-ups, cuyas ecuaciones son

$$x = x_1^2(y_1 + 1) \ , \quad y = x_1(y_1 + 1) \ ,$$

donde (x_1, y_1, z) son las coordenadas en $\mathcal{A}'_1$. Tenemos que $x_1 = f_1$ e $y_1 = g_1$. Además,

$$F = z - x_1(y_1 + 1) - \phi \ ,$$

luego F está γ-preparada (nótese que x_1^2 divide a ϕ). El índice de ramificación de la variable z es ahora 1, por lo tanto el blow-up combinatorio de centro (x_1, z) es un z-paquete de Puiseux $\mathcal{A}'_1 \to \mathcal{A}_1$. Tenemos coordenadas (x_1, y_1, z_1) donde z_1 está dada por

$$z = x_1(z_1 + 1) \ .$$

Se tiene

$$F = x_1(z_1 + \xi_1) - x_1(y_1 + \xi_1) - \phi = x_1(z_1 - y_1 - \frac{\phi}{x_1}) \ .$$

La situación en $\mathcal{A}_1$ es similar a la que teníamos en $\mathcal{A}$, a diferencia de que ahora las variables tienen la mitad de su valor original. Podemos iterar el proceso considerando una sucesión infinita de paquetes de Puiseux

$$\mathcal{A} \xrightarrow{\pi_1} \mathcal{A}'_1 \xrightarrow{\tau_1} \mathcal{A}_1 \xrightarrow{\pi_2} \cdots$$

donde $\pi_i : \mathcal{A}_i \to \mathcal{A}'_{i+1}$ es un y_i-paquete de Puiseux (y de hecho una γ-preparación) y $\tau_i : \mathcal{A}'_i \to \mathcal{A}_i$ es un z_i-paquete de Puiseux. Tenemos que $\mathcal{A}_i = (\mathcal{O}_i, (x_i, y_i, z_i))$, donde $\mathcal{O}_i = \mathbb{C}[z_i, y_i, z_i]_{(x_i,y_i,z_i)}$. Además conocemos los valores de las variables en cualquiera de los modelos $\mathcal{A}_i$ ya que se tiene que $x_i = f_i$, $y_i = g_i$ y $z_i = h_i$. Además, en cada uno de estos modelos la función F está γ-preparada y se escribe de la forma

$$F = x_i^{2^i-1} U_i(z_i - y_i - \sum_{k=1}^{\infty} c_k x_i^{2^i(k-1)+1} V_i) \ ,$$

donde U_i y V_i son unidades de $\mathcal{O}_i = \mathbb{C}[z_s, y_s, z_s]_{(x_i,y_i,z_i)}$. Tenemos por lo tanto que

$$\nu_{\mathcal{A}_i}(F) = \nu(x_i^{2^i-1}) = (2^i - 1)\nu(f_i) = 1 - 2^{-i} < 1 \leq \gamma \ ,$$

luego en ninguno de estos modelos F es γ-final.

Para evitar este fenómeno de acumulación aplicamos un cambio ordenado de coordenadas de tipo Tschirnhaus $\mathcal{A} \to \mathcal{B}$ dado por

$$\tilde{z} = z - y \ .$$

Tenemos que $\nu(\tilde{z}) = \nu(x)$ luego el índice de ramificación de $\tilde{z}$ es 1. La función formal se escribe ahora

$$F = \tilde{z} - \phi \ ,$$

luego esta γ-preparada. Un blow-up de centro $(x, \tilde{z})$ es un $\tilde{z}$-paquete de Puiseux $\mathcal{B} \to \mathcal{B}_1$. En $\mathcal{B}_1$ tenemos coordenadas $(x, y, \tilde{z}_1)$ donde $\tilde{z}_1$ está dada por

$$\tilde{z}_1 = \frac{\tilde{z}}{x} - c_1$$

y tiene valor $\nu(\tilde{z}_1) = 1$. La función F se expresa en estas coordenadas de la forma

$$F = x\left(\tilde{z}_1 - \sum_{k=1}^{\infty} c_{k+1} x^k\right) ,$$

de donde vemos que está γ-preparada. Aplicando $\tilde{z}$-paquetes de Puiseux sucesivamente

$$\mathcal{B} \xrightarrow{\theta_1} \mathcal{B}_1 \xrightarrow{\theta_2} \cdots$$

obtenemos modelos $\mathcal{B}_i$ de coordenadas $(x, y, \tilde{z}_i)$, con $\nu(\tilde{z}_i) = 1$ de forma que F se escribe como

$$F = x^i\left(\tilde{z}_i - \sum_{k=1}^{\infty} c_{k+i} x^k\right) .$$

Tenemos que en cualquiera de estos modelos F está γ-preparada y además

$$\nu_{\mathcal{B}_i}(F) = \nu(x^i) = i .$$

Vemos por tanto que para cualquier índice i mayor que γ la función F es γ-final recesiva en $\mathcal{B}_i$.

En este ejemplo vemos que la función F tiene "valor infinito" respecto a ν. Lo mismo sucede si consideramos la 1-forma con coeficientes formales dF. Además, observamos que el fenómeno de acumulación también puede darse con funciones convergentes: una vez fijado $\gamma \geq 1$ basta considerar la función $F_T = \sum_{k=1}^{T} c_k x^k$ para cualquier $T > \gamma$. Mientras que $\nu(F_T) = T + 1$, mientras realizamos únicamente paquetes de Puiseux su valor explícito será menor que 1. De nuevo, dF_T nos proporciona un ejemplo del mismo fenómeno de acumulación para 1-formas.

- - -

Expliquemos ahora como es el proceso de γ-preparación de una 1-forma $\omega \in N_{\mathcal{A}}^{\ell+1}$ tal que

$$\nu_{\mathcal{A}}(\omega \wedge d\omega) \geq 2\gamma .$$

Descomponemos ω en potencias de z de la forma

$$\omega = \sum_{k=0}^{\infty} z^k \omega_k , \quad \omega_k = \eta_k + f_k \frac{dz}{z} ,$$

donde $\eta_k \in N_{\mathcal{A}}^{\ell}$ y $f_k \in R_{\mathcal{A}}^{\ell}$. Nótese que a diferencia del caso de funciones, los niveles ω_s de una 1-forma $\omega \in N_{\mathcal{A}}^{\ell+1}$ no pertenecen a $N_{\mathcal{A}}^{\ell}$. A cada nivel $\omega_s = \eta_s + f_s dz/z$ le asociamos el par forma-función (η_s, f_s) al cual podemos aplicar $T_5(\ell)$ como explicaremos más adelante.

El valor explícito $\nu_{\mathcal{A}}(\omega_k)$ de un nivel ω_k se define como el mínimo entre $\nu_{\mathcal{A}}(\eta_k)$ y $\nu_{\mathcal{A}}(f_k)$. Dado $\alpha \in \Gamma$, diremos que el nivel ω_k es *α-final dominante* si una de las siguientes condiciones se satisface:

- $\nu_{\mathcal{A}}(\eta_k) < \nu_{\mathcal{A}}(f_k)$ y η_k es α-final dominante.
- $\nu_{\mathcal{A}}(f_k) < \nu_{\mathcal{A}}(\eta_k)$ y f_k es α-final dominante.
- $\nu_{\mathcal{A}}(f_k) = \nu_{\mathcal{A}}(\eta_k)$ y ambos f_k y η_k son α-final dominantes.

El Polígono de Newton Truncado $\mathcal{N}$ y el Polígono de Newton Dominante Truncado Dom$\mathcal{N}$ de una 1-forma se obtienen de la misma manera que los correspondientes a una función: se toma la envolvente positivamente convexa en $\mathbb{R}^2$ de los puntos $(\nu_{\mathcal{A}}(\omega_s), s)$ (todos para $\mathcal{N}$, solamente los $(\gamma - s\nu(z))$-final dominantes para Dom$\mathcal{N}$) junto a $(\gamma, 0)$ y $(0, \gamma/\nu(z))$. De igual modo, diremos que ω está γ-preparada si

$$\mathcal{N} = \mathrm{Dom}\mathcal{N}$$

y además $(\nu_{\mathcal{A}}(\omega_s), s)$ es un punto interior de $\mathcal{N}$ para todo s tal que ω_s no es dominante.

Al igual que en el caso funcional, el primer paso en el proceso de γ-preparación de una 1-forma consiste en aplicar una transformación ℓ-anidada para conseguir que el máximo número posible de niveles ω_s sean $(\gamma - s\nu(z))$-final dominantes. Este paso no precisa de la hipótesis de inducción. Además, la aplicación de una transformación ℓ-anidada no mezcla los niveles entre si, y dentro de cada nivel tampoco interfiere la parte diferencial η_s con la funcional f_s y viceversa.

Tras este primer paso tenemos que Dom$\mathcal{N}$ es estable, es decir, aunque apliquemos más transformaciones ℓ-anidadas el Polígono de Newton Truncado Dominante será el mismo. Tenemos pues que $\mathcal{N} \subset \mathrm{Dom}\mathcal{N}$, y debemos determinar una transformación que haga ambos polígonos iguales (la condición restante para obtener una situación γ-preparada es sencilla una vez que ambos polígonos sean iguales).

Para poder aplicar $T_5(\ell)$ a los niveles ω_s necesitamos conocer el valor explícito $\nu_{\mathcal{A}}(\eta_s \wedge d\eta_s)$. El dato que tenemos es $\nu_{\mathcal{A}}(\omega \wedge d\omega) \geq 2\gamma$. Sin embargo, nada nos garantiza que lo mismo ocurra con las formas η_s. Para saber "cómo de integrable" es cada η_s utilizaremos la siguiente fórmula:

$$\omega \wedge d\omega = \sum_{m=0}^{\infty} z^m \Big(\Theta_m + \frac{dz}{z} \Delta_m \Big)$$

donde

$$\Theta_m := \sum_{i+j=m} \eta_i \wedge d\eta_j$$

y

$$\Delta_m := \sum_{i+j=m} j\eta_j \wedge \eta_i + f_i d\eta_j + \eta_i \wedge df_j \ .$$

Dado que $\nu_{\mathcal{A}}(\omega \wedge d\omega) \geq 2\gamma$ se tiene que

$$\nu_{\mathcal{A}}(\Theta_m) \geq 2\gamma \quad \text{y} \quad \nu_{\mathcal{A}}(\Delta_m) \geq 2\gamma$$

para todo $m \geq 0$. Por otro lado, para cualquier 1-forma σ se tiene

$$\nu_A(d\sigma) \geq \nu_A(\sigma) \ .$$

Gracias a esta observación, tenemos que $\nu_{\mathcal{A}}(\Theta_{2s}) \geq 2\gamma$ implica

$$\nu_{\mathcal{A}}(\eta_s \wedge d\eta_s) \geq \min \big\{ \{2\gamma\} \cup \{\nu_{\mathcal{A}}(\eta_{s-i}) + \nu_{\mathcal{A}}(\eta_{s+i})\}_{1 \leq i \leq s} \big\} \ .$$

Ésta fórmula nos permite aplicar de forma iterada $T_5(\ell)$ comenzando por el nivel más bajo no dominante y continuando con los niveles superiores hasta alcanzar la cota marcada por $\gamma/\nu(z)$. Tras ello, recomenzamos por el nivel más bajo de nuevo y ascendemos hasta dicha cota. Este proceso nos permite aproximar $\mathcal{N}$ a $\mathrm{Dom}\mathcal{N}$ hasta una distancia prefijada como se indica en el Lema **??**. De este modo, podremos conseguir en particular que todos los vértices de $\mathrm{Dom}\mathcal{N}$ sean a su vez vértices de $\mathcal{N}$. Este procedimiento es esencial pero no suficiente para completar la γ-preparación.

Tras esta aproximación de $\mathcal{N}$ a $\mathrm{Dom}\mathcal{N}$ completamos la γ-preparación usando resultados de "proporcionalidad truncada" como la siguiente versión truncada del Lema de De Rham-Saito ([20]):

> Sean η y σ dos 1-formas. Si η es log-elemental y $\nu_{\mathcal{A}}(\eta \wedge \sigma) = \alpha$ entonces existen una función f y una 1-forma $\bar{\sigma}$ con $\nu_{\mathcal{A}}(\sigma) > \alpha$ tales que $\sigma = f\eta + \bar{\sigma}$.

\- - -

Una vez completado el proceso de γ-preparación, al igual que en el caso de funciones, tomamos la altura crítica χ de $\mathcal{N}$ como invariante de control principal. Debemos ahora analizar el comportamiento de χ tras aplicar un z-paquete de Puiseux, con función racional de contacto $z^d/\boldsymbol{x}^{\boldsymbol{p}}$, seguido de una nueva γ-preparación. Excepto en el caso de que se den alguna de las *condiciones de resonancia* (R1) o (R2) definidas en el Capítulo 7 la altura crítica descenderá. Si en algún momento alcanzamos la situación $\chi = 0$, un z-paquete de Puiseux y una γ-preparación más nos permitirán obtener una situación γ-final.

La condición (R1) requiere que el exponente de ramificación d sea estrictamente mayor que 1. Como demostraremos, esta condición, de suceder, no puede repetirse. Por su parte, la condición (R2) requiere que el exponente de ramificación sea $d = 1$, pero a diferencia de la anterior, ésta si se puede dar de forma consecutiva indefinidas veces.

Por lo tanto, resta considerar el caso en el que la condición (R1) se da indefinidamente. En esta situación, cada z-paquete de Puiseux hace aumentar el valor explícito de ω. Si éste llega a sobrepasar γ habremos alcanzado un modelo en el que ω es γ-final recesiva. Sin embargo, al igual que en el caso funcional, puede ocurrir que dicho valor se "acumule". Para evitar este fenómeno, recurriremos a un cambio de coordenadas de tipo Tschirnhaus. La determinación de dicho cambio de coordenadas se hará gracias a las propiedades de proporcionalidad truncada similares a las utilizadas en la última fase del proceso de γ-preparación.

Introduction

This work is devoted to show the following statement

Theorem I. *Let k be a field of characteristic zero and let K/k be a finitely generated field extension. Let $\mathcal{F}$ be a rational codimension one foliation of K/k. Given a k-rational archimedean valuation ν of K/k, there is a projective model M of K/k such that $\mathcal{F}$ is log-final at the center of ν in M.*

The proof of the theorem follows the classical ideas of Zariski's local uniformization, working by induction on the number of variables. The main difficulty when dealing with integrable 1-forms is that the full integrability property is lost during the induction process. In order to solve it, we have structured our results in terms of "valuated truncations" of formal functions and differential 1-forms. An important part of our work consists in the control of a partial integrability condition stable by the truncation process.

Theorem I is a first step in the proof of the following conjecture, whose complete poof is the goal of future works:

Conjecture. *Let k be a field of characteristic zero and let K/k be a finitely generated field extension. Let $\mathcal{F}$ be a rational codimension one foliation of K/k. Given a valuation ν of K/k, there is a projective model M of K/k such that $\mathcal{F}$ is log-final at the center of ν in M.*

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Consider a finitely generated field extension K/k with transcendence degree $\operatorname{tr.deg}(K/k) = n$ over an algebraically closed zero characteristic field k. Let $\{z_1, z_2, \ldots, z_n\} \subset K$ be a transcendence basis of such field extension. We have the following tower of fields

$$k \subset k(z_1, z_2, \ldots, z_n) \subset K \ ,$$

where K is a finitely generated algebraic field extension (thus separable since $\operatorname{char}(k) = 0$) of $k(z_1, z_2, \ldots, z_n)$. The module of Kähler differentials $\Omega_{K/k}$ is a K-vector space of dimension $n = \operatorname{tr.deg}(K/k)$ and $\{dz_1, dz_2, \ldots, dz_n\}$ is a basis of such space. A *rational codimension one foliation $\mathcal{F}$ of K/k* is a K-vector subspace of dimension one of $\Omega_{K/k}$ such that for every 1-form $\omega \in \mathcal{F}$ the integrability condition

$$\omega \wedge d\omega = 0$$

is satisfied.

This definition of foliation agrees with the classical definition in complex projective geometry. Consider the complex projective space $\mathbb{P}^n_{\mathbb{C}}$ and a affine cover $\mathbb{P}^n_{\mathbb{C}} = U_0 \cup U_1 \cup \cdots \cup U_n$. A codimension one foliation of $\mathbb{P}^n_{\mathbb{C}}$ is given by $n+1$ integrable homogeneous polynomial 1-forms

$$W_i = \sum_{j=1}^{n} P^i_j(z^i_1, z^i_2, \ldots, z^i_n)\, dz^i_j \ , \quad i = 0, 1, \ldots, n \ ,$$

defined over the affine charts $U_i \simeq \mathbb{C}[z^i_1, z^i_2, \ldots, z^i_n]$, in such a way that

$$W_{i|_{U_i \cap U_j}} = G_{ij}\, W_{\ell|_{U_i \cap U_j}} \ ,$$

where G_{ij} is an invertible rational function $U_i \cap U_j$. The field of rational functions of any affine chart U_i and $\mathbb{P}^n_{\mathbb{C}}$ itself, is

$$K \simeq \mathbb{C}(z^i_1, z^i_2, \ldots, z^i_n)$$

for any index i. All the 1-forms W_i can be considered as elements of $\Omega_{K/\mathbb{C}}$. Any of them spans the same vector subspace of dimension 1

$$\mathcal{F} =< W_i >\subset \Omega_{K/\mathbb{C}} \ ,$$

which is a codimension one rational foliation of $K/\mathbb{C}$ following ou definition.

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A projective model of K/k is a projective k-variety M, in the sense of scheme theory, such that $K = \kappa(M)$ is its field of rational functions. Take a regular k-rational point $Y \in M$, it means, a point such that the local ring $\mathcal{O}_{M,Y}$ is regular and its residue field is $\kappa_{M,Y} \simeq k$. A system of generators $z_1, z_2, \ldots, z_n$ of the maximal ideal $\mathfrak{m}_{M,Y}$ is also a transcendence basis of K/k, thus it provides a basis $dz_1, dz_2, \ldots, dz_n$ of $\Omega_{K/k}$. Consider a system of generators of $\mathfrak{m}_{M,Y}$ of the form $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y})$, where $\boldsymbol{x} = (x_1, x_2, \cdots, x_r)$ and $\boldsymbol{y} = (y_1, y_2, \cdots, y_{n-r})$. Let $\Omega_{\mathcal{O}_{M,Y}/k}(\log \boldsymbol{x})$ be the $\mathcal{O}_{M,Y}$-submodule of $\Omega_{K/k}$ generated by $\Omega_{\mathcal{O}_{M,Y}/k}$ and the logarithmic differentials

$$\frac{dx_1}{x_1}, \frac{dx_2}{x_2}, \ldots, \frac{dx_r}{x_r} \ .$$

We have that $\Omega_{\mathcal{O}_{M,Y}/k}(\log \boldsymbol{x})$ is a free $\mathcal{O}_{M,Y}$-module of rank n generated by

$$\frac{dx_1}{x_1}, \frac{dx_2}{x_2}, \ldots, \frac{dx_r}{x_r}, dy_1, dy_2, \ldots, dy_{n-r} \ .$$

Let $\mathcal{F}$ be a codimension one rational foliation of K/k. Consider

$$\mathcal{F}_{M,Y}(\log \boldsymbol{x}) = \mathcal{F} \cap \Omega_{\mathcal{O}_{M,Y}/k}(\log \boldsymbol{x}) \ .$$

$\mathcal{F}_{M,Y}(\log \boldsymbol{x})$ is a free $\mathcal{O}_{M,Y}$-module of rank 1 generated by an integrable 1-form

$$\omega = \sum_{i=1}^{r} a_i \frac{dx_i}{x_i} + \sum_{j=1}^{n-r} b_j dy_j \ ,$$

where the coefficients $a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_{n-r} \in \mathcal{O}_{M,Y}$ have no common factor. We say that

1. $\mathcal{F}$ is $\boldsymbol{x}$*-log elementary* at $Y \in M$ if $(a_1, a_2, \ldots, a_r) = \mathcal{O}_{M,Y}$;
2. $\mathcal{F}$ is $\boldsymbol{x}$*-log canonical* at $Y \in M$ if $(a_1, a_2, \ldots, a_r) \subset \mathfrak{m}_{M,Y}$ and in addition

$$(a_1, a_2, \ldots, a_r) \not\subset (x_1, x_2, \ldots, x_r) + \mathfrak{m}_{M,Y}^2 \ .$$

We say that $\mathcal{F}$ is *x-log final* at $Y \in M$ if it is $\boldsymbol{x}$-log elementary or $\boldsymbol{x}$-log canonical. Finally, we say that $\mathcal{F}$ is *log-final* at $Y \in M$ if it is $\boldsymbol{x}$-log final for certain system of generators $(\boldsymbol{x}, \boldsymbol{y})$ of $\mathfrak{m}_{M,Y}$.

To be log-final is the algebraic version of the concept of pre-simple singularity of the complex analytic case ([21],[7],[5]). Let us briefly recall this definition. Consider a foliation of $(\mathbb{C}^2, \mathbf{0})$ given locally by

$$a(x,y)dx + b(x,y)dy = 0 \ .$$

The origin $(0,0)$ is a pre-simple singularity if the foliation is non singular (one of the series $a(x,y)$ or $b(x,y)$ is a unit) or if the Jacobian matrix

$$\begin{pmatrix} \partial b/\partial x(0,0) & -\partial a/\partial x(0,0) \\ \partial b/\partial y(0,0) & -\partial a/\partial y(0,0) \end{pmatrix}$$

is non-nilpotent. In this situation we can always take local analytic coordinates x', y' such that the foliation is locally given by

$$a'(x',y')\frac{dx'}{x'} + b'(x',y')dy' = 0 \ ,$$

where $a'(x',y') = y' + \cdots$, thus the foliation is x'-log-final at the origin, with respect to the local analytic coordinates (x',y'). A more detailed study can be found in [8]. In general, for foliations over complex ambient spaces of arbitrary dimension, the concept of pre-simple singularity introduced in [7] and [5] is equivalent to the property of being log-final.

In the case of foliations over algebraic varieties of dimension 2 or 3, the theorem proved in this work, as well as the conjecture, are consequences of the following global results of reduction of singularities (see [21] for the two-dimensional case and [5] for the dimension 3):

Theorem (A. Seidenberg, 1968; F. Cano 2004)**.** *Let $\mathcal{F}$ be a codimension one rational foliation of $(\mathbb{C}^n, 0)$, $n = 2, 3$. There is a finite composition of blow-ups*

$$(\mathbb{C}^n, 0) \leftarrow (M_1, Z_1) \leftarrow \cdots \leftarrow (M_N, Z_N) = (M, Z)$$

such that $\mathcal{F}$ is log-final at every point $Y \in Z$.

In the case of ambient spaces o dimension $n \geq 4$ the global reduction of singularities of codimension one foliations is an open problem.

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Resolution of singularities of algebraic varieties over a ground field of characteristic 0 was achieved by Hironaka in its celebrated paper [14].

Theorem (Hironaka's Reduction of Singularities, 1964)**.** *Let K/k be a finitely generated field extension, where k has characteristic zero. There is a non-singular projective model M of K/k.*

Before the work of Hironaka, the problem had been solved for dimension at most three. The case of complex curves was already treated by Newton in 1676. For surfaces, the first rigorous proof is due to Walker in 1935 [25]. The case of 3-folds was solved by Zariski in 1944 [27]. Before this result, Zariski proved local uniformization for algebraic varieties in characteristic zero in arbitrary dimension [26].

Theorem (Zariski's Local Uniformization, 1940)**.** *Let K/k be a finitely generated field extension, where k has characteristic zero and consider a valuation ν of K/k. There is a projective model M of K/k such that the center of ν in M is a regular point.*

In [28], Zariski showed the compactness of the Zariski-Riemann space (the space of valuations of K/k), which implies that a finite number of projective models are enough to support local uniformizations for any valuation. Then, he obtained his global result by patching projective models, but this method only works in dimension at most 3.

The general problems of resolution of singularities has a long history after the works of Zariski and Hironaka. The case of complex analytic spaces was achieved by Aroca, Hironaka and Vicente [3]. The huge original proofs of Hironaka have been analyzed very carefully with emphasis in the constructiveness and functorial properties by Villamayor [24], Bierstone and Milman [4] and others.

One of the keys of the resolution of singularities is the maximal contact theory and its differential version, developed by Giraud [13]. This is one of the starting points of the strongest known results in positive characteristic, due to Cossart and Piltant, who proved resolution of singularities of 3-folds in positive characteristic [10, 11]. This work improves the results of Abhyankar, which shows resolution in positive characteristic for surfaces [1], and for 3-folds in the case of fields of characteristic at least 7 [2]. All of these results in positive characteristic are obtained passing through local uniformization.

Another related problems are concerning the monomialization of morphisms in zero characteristic due to Cutkosky [12]. The difficulties in this case are close to the ones for case of vector fields or foliations.

Reduction of singularities of vector fields in dimension two was achieved by Seidenberg [21]. In dimension 3, there are partial results due to Cano [6], an then this author, Roche and Spivakovsky obtain a global reduction of singularities through local uniformization [9], using the axiomatic Zariski's patching method developed by Piltant [18]. Recently McQuillan and Panazzolo have treated the 3-dimensional case from a non-birational viewpoint [17].

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In this work we treat the case of k-rational rank one valuations of K/k. In the classical Zariski's approach to the Local Uniformization problem, this is the starting point, and it concentrates the main algorithmic and combinatoric difficulties. We hope the situation to be similar in the case of codimension one foliations, and that starting from the result obtained in this work we can complete the proof of the conjecture in future works.

A valuation $\nu : K^* \to \Gamma$ of K/k is k-rational if its residue field κ_ν is isomorphic to the base field k. The valuation ν is archimedean if and only if there is an inclusion of ordered groups $\Gamma \subset (\mathbb{R}, +)$.

Let M be a projective model of K/k. The center of ν at M is the unique point $Y \in M$ such that for every $\phi \in \mathcal{O}_{M,Y}$ we have

$$\nu(\phi) \geq 0 \quad \text{and} \quad \nu(\phi) > 0 \Leftrightarrow \phi \in \mathfrak{m}_{M,Y} \ .$$

Such a point always exists and it is unique (see [26] or [19]). In addition, there is a tower of fields

$$k \subset \kappa_{M,Y} \subset \kappa_\nu \ .$$

Since we only consider k-rational valuations we have that $k = \kappa_{M,Y}$ and therefore the centers of ν in each projective model are k-rational points (in particular they are closed points).

The *rational rank* $\operatorname{rat.rk}(\nu)$ is the dimension over $\mathbb{Q}$ of $\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$. Abhyankar's inequality guarantees that $\operatorname{rat.rk}(\nu) \leq \operatorname{tr.deg}(K/k)$. The rational ranks correspond with the maximum number of elements $\phi_1, \phi_2, \ldots, \phi_r \in K^*$ with $\mathbb{Z}$-independents values $\nu(\phi_1), \nu(\phi_2), \ldots, \nu(\phi_r) \in \Gamma$.

Our technical results are stated in terms of *parameterized regular local models*. A parameterized regular local model $\mathcal{A}$ for $K/k, \nu$ is a pair

$$\mathcal{A} = (\mathcal{O}, (\boldsymbol{x}, \boldsymbol{y}))$$

such that

1. There is a projective model M of K/k such that the center Y of ν is a regular point of M and $\mathcal{O} = \mathcal{O}_{M,Y}$;

2. The list $(\boldsymbol{x}, \boldsymbol{y}) = (x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_{n-r})$, where $r = \operatorname{rat.rk}(\nu)$, is a regular system of parameters of $\mathcal{O}$ and the values $\nu(x_1), \nu(x_2), \ldots, \nu(x_r)$ are $\mathbb{Z}$-independent.

The existence of parameterized regular local models is proved using the global resolution of singularities of Hironaka [14]. This proof can be found in [9], where such models are introduced.

According with this terminology, given a rational codimension one foliation $\mathcal{F}$ of K/k, we denote

$$\mathcal{F}_{\mathcal{A}} = \mathcal{F} \cap \Omega_{\mathcal{O}/k}(\log \boldsymbol{x}) = \mathcal{F}_{M,Y}(\log \boldsymbol{x}) \ .$$

We say that $\mathcal{F}$ is $\mathcal{A}$-*final* if $\mathcal{F}_{\mathcal{A}}$ is $\boldsymbol{x}$-log final.

We will use transformations $\mathcal{A} \to \mathcal{A}'$ of parameterized regular local models, called *basic operations*. Such transformations have an underlying morphism $\mathcal{O} \to \mathcal{O}'$ which can be either a blow-up or the identity morphism. Let us describe them:

- *Ordered coordinate changes*. The underlying morphism $\mathcal{O} \to \mathcal{O}'$ is the identity. Given an index $0 \leq \ell \leq n-r$ we consider a new coordinate y'_ℓ given by
$$y'_\ell = y_\ell + \psi(\boldsymbol{x}, y_1, y_2, \ldots, y_{\ell-1}),$$
where $\psi(\boldsymbol{x}, y_1, y_2, \ldots, y_{\ell-1}) \in k[\boldsymbol{x}, y_1, y_2, \ldots, y_{\ell-1}]$ is written as
$$\psi(\boldsymbol{x}, y_1, y_2, \ldots, y_{\ell-1}) = \sum_I \boldsymbol{x}^I \psi_I(y_1, y_2, \ldots, y_{\ell-1})$$
with $\nu(\boldsymbol{x}^I) \geq \nu(y_\ell)$ if $\psi_I \neq 0$.

- *Blow-ups of codimension two centers.* The center of the blow-up will be either $x_i = x_j = 0$ or $x_i = y_j = 0$. The ring $\mathcal{O}'$ is

  $$\mathcal{O}' = \mathcal{O}[\boldsymbol{x}', \boldsymbol{y}']_{(\boldsymbol{x}', \boldsymbol{y}')}$$

  where the coordinates $(\boldsymbol{x}', \boldsymbol{y}')$ are given by:

  1. If the center is $x_i = x_j = 0$ and in addition $\nu(x_i) < \nu(x_j)$, then $x'_j := x_j/x_i$.
  2. If the center is $x_i = y_j = 0$ and in addition $\nu(x_i) < \nu(y_j)$, then $y'_j := y_j/x_i$.
  3. If the center is $x_i = y_j = 0$ and in addition $\nu(x_i) > \nu(y_j)$, then $x'_i := x_i/y_j$.
  4. If the center is $x_i = y_j = 0$ and in addition $\nu(x_i) = \nu(y_j)$, then $y'_j := y_j/x_i - \xi$, where $\xi \in k^*$ is the unique constant such that $\nu(y_j/x_i - \xi) > 0$.

  The last case is a *blow-up with translation.* The remaining cases are *combinatorial blow-ups.*

Theorem I is a consequence of the following statement in terms of parameterized regular local models:

Theorem II. *Let k be a field of characteristic zero and let K/k be a finitely generated field extension. Let $\mathcal{F}$ be a rational codimension one foliation of K/k. Given a k-rational archimedean valuation ν of K/k and a parameterized regular local model $\mathcal{A}$ of $K/k, \nu$, there is a finite composition of basic transformations*

$$\mathcal{A} = \mathcal{A}_0 \to \mathcal{A}_1 \to \cdots \to \mathcal{A}_N = \mathcal{B}$$

such that $\mathcal{F}$ is $\mathcal{B}$-final.

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We systematically consider the formal completion $\widehat{\mathcal{O}}$ of the local ring $\mathcal{O}$. A first reason for doing that is of practical nature, since

$$\widehat{\mathcal{O}} = k[[\boldsymbol{x}, \boldsymbol{y}]],$$

we can consider the elements of $\mathcal{O}$ as being formal series. The second reason to consider the formal completion is due to the fact that the solutions of differential equations with coefficients in $\mathcal{O}$ need not to be in the same ring $\mathcal{O}$ (even in the case that we are working in the analytic category).

Let us illustrate this with an example. If our "problem object" is a function $f \in \mathcal{O}$, after finitely many basic transformations we obtain a parameterized regular local model $\mathcal{A}' = (\mathcal{O}', (\boldsymbol{x}', \boldsymbol{y}'))$ such that

$$f = \boldsymbol{x}'^{\boldsymbol{p}} U \ , \quad U \in \mathcal{O}' \setminus \mathfrak{m}' \ .$$

This is a direct consequence of Zariski's Local Uniformization. For completeness we include a proof for the case of functions which we will use as a guide for the case of differential 1-forms. If we consider the foliation given by $df = 0$ we have that it is $\mathcal{A}'$-final, in particular it is $\boldsymbol{x}$-log elementary. This property is always satisfied by foliations having a first integral: it is always possible to reach a model in which the foliation is $\boldsymbol{x}$-log elementary. However, this does not happen in general. In dimension two we have an example given by Euler's Equation:

$$(y-x)\frac{dx}{x} - xdy = 0 \ .$$

The foliation of $(\mathbb{C}^2, \mathbf{0})$ given by this equation is x-log canonical. In addition, it has an invariant formal curve with equation $\hat{y} = 0$ where

$$\hat{y} = y - \sum_{n=0}^{\infty} n!\, x^{n+1} \ .$$

We have the equality

$$(y-x)\frac{dx}{x} - xdy = \hat{y}\left(\frac{dx}{x} - x\frac{d\hat{y}}{\hat{y}}\right) \ ,$$

thus if we allow the use of formal coordinates $(x, \hat{y})$, the foliation is given by

$$\frac{dx}{x} - x\frac{d\hat{y}}{\hat{y}} = 0 \ ,$$

hence it would be "$x\hat{y}$-log elementary". If we consider the valuation of $\mathbb{C}(x,y)/\mathbb{C}$ given by

$$\nu(f(x,y)) = \mathrm{ord}_t(f(t, \sum_{n=0}^{\infty} n! t^{n+1}))$$

we can check that it is not possible to reach by means of basic transformations a parameterized regular local model in which the foliation is x-log elementary. Regardless of the basic transformations we perform the foliation will be x-log canonical. In fact, ν is the only valuation of $\mathbb{C}(x,y)/\mathbb{C}$ which satisfies this property. This is due to the fact that ν follows the infinitely near points of the invariant formal curve $\hat{y} = 0$.

In some sense, we can think that the differential 1-form $\omega = xdy - (y-x)dx/x$ has "infinite value" with respect to ν. The property of 'having infinite value" can be also satisfied by formal functions $\hat{f} \in \hat{\mathcal{O}} \setminus \mathcal{O}$ (in this example $\hat{y} \in \hat{\mathcal{O}}$ has "infinite value"), but it can never be satisfied by elements $f \in \mathcal{O} \setminus \{0\}$. Note that although ω has "infinite value" its coefficients do not have it.

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One of the main differences of our procedure with respect to the classical Zariski's approach to Local Uniformization is that we proceed by systematically considering truncations relative to a given element of the value group. This method is essential for us since thanks to it we can control the integrability condition inside the general induction procedure. In addition, this method can be applied to formal functions as well as differential 1-forms with formal coefficients.

Let $\mathcal{A} = \big(\mathcal{O}; (\boldsymbol{x}, \boldsymbol{y})\big)$ be a parameterized regular local model for K, ν, where $\boldsymbol{x} = (x_1, x_2, \ldots, x_r)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_{n-r})$. For each index $0 \le \ell \le n - r$, consider the power series ring

$$R^{\ell}_{\mathcal{A}} := k[[\boldsymbol{x}, y_1, y_2, \ldots, y_\ell]] \ .$$

Note that $R_{\mathcal{A}}^{n-r} \simeq \hat{\mathcal{O}}$. For each index $0 \leq \ell \leq n-r$ we define $N_{\mathcal{A}}^{\ell}$ as the $R_{\mathcal{A}}^{\ell}$-module generated by

$$\frac{dx_1}{x_1}, \frac{dx_2}{x_2}, \ldots, \frac{dx_r}{x_r}, dy_1, dy_2, \ldots, dy_\ell \ .$$

Any element $\omega \in N_{\mathcal{A}}^{\ell}$ may be written in a unique way as $\omega = \sum_I x^I \omega_I$, where

$$\omega_I = \sum_{i=1}^{r} a_{I,i}(y_1, y_2, \ldots, y_\ell) \frac{dx_i}{x_i} + \sum_{j=1}^{\ell} b_{I,j}(y_1, y_2, \ldots, y_\ell) dy_j \ .$$

We define the *explicit value* $\nu_{\mathcal{A}}(\omega)$ to be the minimum among the values $\nu(x^I)$ such that $\omega_I \neq 0$.

Let us fix an element of the value group $\gamma \in \Gamma$. Take $\omega \in N_{\mathcal{A}}^{\ell}$ and let x^{I_0} be the monomial such that $\nu_{\mathcal{A}}(\omega) = \nu(x^{I_0})$. We say that ω is *γ-final in $\mathcal{A}$* if one of the following situations holds

- *γ-final dominant case:* $\nu_{\mathcal{A}}(\omega) \leq \gamma$ and at least one of the coefficients $a_{I_0,i}$ satisfies $a_{I_0,i}(0) \neq 0$ (equivalently if ω_{I_0} is x-log elementary).
- *γ-final recessive case:* $\nu_{\mathcal{A}}(\omega) = \nu(x^{I_0}) > \gamma$.

Theorem II (thus Theorem I too) is a consequence of the following result stated in terms of truncations:

Theorem III (Truncated Local Uniformization)**.** *Let $\omega \in N_{\mathcal{A}}^{\ell}$ be a 1-form and fix a value $\gamma \in \Gamma$. If*

$$\nu_{\mathcal{A}}(\omega \wedge d\omega) \geq 2\gamma \ ,$$

then there is a finite composition of basic transformations $\mathcal{A} \to \mathcal{B}$, which do not affect to the variables y_j with $j > \ell$, such that ω is γ-final in $\mathcal{B}$.

The main and more technical part of this work, Chapters 6 and 7, is devoted to prove Theorem III. In Chapter 8 we show how to derive Theorem I from Theorem III.

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We prove Theorem III by induction on the number of dependent variables ℓ. Instead of perform arbitrary compositions of basic transformations, we will restrict ourselves to the *ℓ-nested transformations* which we define by induction. A *0-nested transformation* is a finite composition of blow-ups of the kind $x_i = x_j = 0$ (in particular all oh them are combinatorial). A *ℓ-nested transformation* is a finite composition of $(\ell-1)$-nested transformations, *ℓ-Puiseux's packages* and ordered changes of the ℓ-th variable. These kind of transformations are closely related with the *monoidal transform sequences* and the *uniformizing transform sequences* used by Cutkosky in [12].

Given a dependent variable y_ℓ, its value linearly depends on the value of the independent variables $\boldsymbol{x}$. It means that there are integers $d \geq 1$ and $p_1, \ldots, p_r$ such that

$$d\nu(y_\ell) = p_1\nu(x_1) + \cdots + p_r\nu(x_r) \ ,$$

or equivalently, the *contact rational function* $y_\ell^d/\boldsymbol{x}^{\boldsymbol{p}}$ has zero value. Since the valuation is k-rational, there is a unique constant $\xi \in k^*$ such that $y_\ell^d/x^p - \xi$ has positive value. A ℓ-*Puiseux's package* $\mathcal{A} \to \mathcal{A}'$ is any finite composition of blow-ups of parameterized regular local models with centers of the kind $x_i = x_j = 0$ or $x_i = y_\ell = 0$ such that all the blow-ups are combinatorial except the last one. In particular, we have that $y'_\ell = y_\ell^d/\boldsymbol{x}^{\boldsymbol{p}} - \xi$. The existence of Puiseux's packages follows form the resolution of singularities of the binomial ideal $(y_\ell^d - \xi \boldsymbol{x}^{\boldsymbol{p}})$.

The Puiseux's packages were introduced in [9] for the treatment of vector fields. In the two-dimensional case, the Puiseux's packages are directly related with the Puiseux's pairs of the analytic branches that the valuation follows.

The statements $T_3(\ell)$, $T_4(\ell)$ and $T_5(\ell)$ formulated below are proved by induction on ℓ. The statement $T_3(\ell)$ is about 1-forms with formal coefficients. In particular, Theorem III is equivalent to $T_3(n-r)$. The statement $T_4(\ell)$ is about formal functions, while $T_5(\ell)$ deals with pairs function-form - in fact this result can be state and prove in a more general way for finite lists os functions and 1-forms without adding difficulties, but is this precise formulation which we will use inside the induction procedure -.

The concept of explicit value previously defined for 1-forms extends directly to the case of functions, pairs function-form and p-forms for any $p \geq 2$. In the same way, given a value $\gamma \in \Gamma$ we extend the definition of γ-final 1-forms to the case of functions and pairs function-form.

Let $F \in R^\ell_{\mathcal{A}}$ be a formal function and let $\omega \in N^\ell_{\mathcal{A}}$ be a 1-form. We state the following results:

$\boldsymbol{T_3(\ell)}$ **:** Assume that $\nu_{\mathcal{A}}(\omega \wedge d\omega) \geq 2\gamma$. There is a ℓ-nested transformation $\mathcal{A} \to \mathcal{B}$ such that ω is γ-final in $\mathcal{B}$.

$\boldsymbol{T_4(\ell)}$ **:** There is a ℓ-nested transformation $\mathcal{A} \to \mathcal{B}$ such that F is γ-final in $\mathcal{B}$.

$\boldsymbol{T_5(\ell)}$ **:** Assume that $\nu_{\mathcal{A}}(\omega \wedge d\omega) \geq 2\gamma$. There is a ℓ-nested transformation $\mathcal{A} \to \mathcal{B}$ such that the pair (ω, F) is γ-final in $\mathcal{B}$.

The statements $T_3(0)$, $T_4(0)$ and $T_5(0)$ can be proved easily by means of the control of the Newton Polyhedron of an ideal under combinatorial blow-ups [23].

As induction hypothesis we assume that statements $T_3(k)$, $T_4(k)$ and $T_5(k)$ are true for all $k \leq \ell$. We will see that $T_3(\ell+1)$ implies $T_4(\ell+1)$ and $T_5(\ell+1)$, although in Chapter 5 we include the proof of $T_4(\ell+1)$ since we will use this proof as a guide for the case of differential 1-forms. The more difficult step is to prove $T_3(\ell+1)$ making use of the hypothesis induction.

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Assuming the hypothesis induction, we divide the proof of $T_3(\ell+1)$ in two steps:

1. Process of γ-*preparation* of a 1-form $\omega \in N^{\ell+1}_{\mathcal{A}}$ by means of ℓ-nested transformations (Chapter 6).

2. Getting γ-final 1-forms. We define invariants related to a γ-prepared 1-form $\omega \in N^{\ell+1}_{\mathcal{A}}$ and by controlling them we determine a $(\ell+1)$-nested transformation in such a way that ω is γ-final in the parameterized regular local model obtained (Chapter 7).

We end this introduction with a brief description of these steps.

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As an example, we show how to obtain Local Uniformization of a function using the method of truncations. The difficulties encountered in the process of γ-preparation of a 1-form do not appear in the case of functions. The main difference lies in the nature of the objects to which we apply the induction hypothesis. Given a formal function $F \in R_{\mathcal{A}}^{\ell+1}$ we can write it as a power series in the last dependent variable

$$F = \sum_{s \geq 0} y_{\ell+1}^s F_s(\boldsymbol{x}, y_1, y_2, \ldots, y_\ell) \ .$$

The coefficients F_s belong to $R_{\mathcal{A}}^{\ell}$ thus we can apply $T_4(\ell)$ with them. However, given a 1-form $\omega \in N_{\mathcal{A}}^{\ell+1}$, if we "decompose" it in the same way

$$\omega = \sum_{s \geq 0} y_{\ell+1}^s \omega_s(\boldsymbol{x}, y_1, y_2, \ldots, y_\ell) \ ,$$

the "coefficients" ω_s do not belong to $N_{\mathcal{A}}^{\ell}$, so we can not use $T_3(\ell)$ directly.

Since we proceed by induction on the parameter ℓ we denote $z = y_{\ell+1}$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_\ell)$. As before, given $F \in R_{\mathcal{A}}^{\ell+1}$ we consider the decomposition $F = \sum z^s F_s$. We define the *Truncated Newton Polygon* $\mathcal{N}(F; \mathcal{A}; \gamma)$ as the positive convex hull in $\mathbb{R}^2$ of the points $(0, \gamma/\nu(z))$, $(\gamma, 0)$ and $(\nu_{\mathcal{A}}(F_s), s)$ for $s \geq 0$. In the same way we define the*Truncated Dominant Newton Polygon* Dom$\mathcal{N}$, this time considering $(0, \gamma/\nu(z))$, $(\gamma, 0)$ and only the points $(\nu_{\mathcal{A}}(F_s), s)$ such that F_s is $(\gamma - s\nu(z))$-final dominant. The function F is γ-prepared in $\mathcal{A}$ if

$$\mathcal{N} = \mathrm{D}om\mathcal{N}$$

and in addition $(\nu_{\mathcal{A}}(F_s), s)$ is an interior point of $\mathcal{N}$ for any non dominant level F_s.

The γ-preparation of a function has no major difficulties. First, since a ℓ-nested transformation does not mix the levels F_s between them, we can perform a ℓ-nested transformation in such a way that we obtain the maximum number of $(\gamma - s\nu(z))$-final levels F_s (this operation does not use of the induction hypothesis). Then, the induction hypothesis allow us to reach a γ-prepared situation directly: it is enough to apply $T_4(\ell)$ to the levels F_s with $s \leq \gamma/\nu(z)$ which are non-dominant.

Once we have a γ-prepared situation, the *critical height* χ is well defined. It corresponds with the highest point of the side of $\mathcal{N}$ with slope $-1/\nu(z)$ (see Section 5.2).

Now we must to study the behavior of χ after performing a z-Puiseux's package with contact rational function $z^d/\boldsymbol{x}^{\boldsymbol{p}}$, followed by a new γ-preparation.

If the *ramification index* d is strictly greater than 1 we obtain $\chi' < \chi$. If at one moment we have that the critical height is 0, after one more z-Puiseux's package and a γ-preparation we reach a γ-final situation.

Therefore, it only remains to consider the case $d = 1$ and $\chi > 0$ indefinitely. In this situation, each z-Puiseux's package increases the explicit value of F. If a one moment such value becomes greater than γ we have reach a γ-final recessive situation. However, it may happen that the value "accumulates" before reaching γ. This phenomenon is due to the possibility of having $d \geq 2$ before the γ-preparation, but $d = 1$ after performing it. In this situation we use a partial

Tschirnhausen transformation (a certain kind of ordered change of coordinates). In the case of differential 1-forms we also use this kind of change of variables, but its determination is more subtle.

Let us illustrate this situation with an example. Define recursively the following elements of $\mathbb{C}(x,y,z)$ for $j \geq 0$:

$$f_{j+1} = \frac{f_j}{g_j} \ , \quad g_{j+1} = \frac{g_j^2}{f_j} - 1 \ , \quad h_{j+1} = \frac{g_j h_j}{f_j} - 1 \ ,$$

where

$$f_0 = x \ , \quad g_0 = y \ , \quad h_0 = z \ .$$

Let ν be a valuation of $\mathbb{C}(x,y,z)/\mathbb{C}$ such that

$$\nu(x) = 1 \ , \quad \nu(y) = \nu(z) = \frac{1}{2} \ , \quad \nu(y-z) = 1 \ ,$$

and in addition there is a transcendental series $\phi = \sum_{k=1}^{\infty} c_k x^k$ such that

$$\nu\left(y - z - \sum_{k=1}^{t} c_k x^k\right) = t+1 \quad \text{for all } t \geq 1 \ .$$

We have that $\mathcal{A} = (\mathcal{O}, (x,y,z))$ where $\mathcal{O} = \mathbb{C}[z,y,z]_{(x,y,z)}$ is a parameterized regular local model of $\mathbb{C}(x,y,z)/\mathbb{C}, \nu$. Using the recurrence relations we conclude that for any $j \geq 0$ we have

$$\nu(f_j) = 2^{-j} \ , \quad \nu(g_j) = \nu(h_j) = 2^{-j-1} \quad \text{and} \quad \nu(g_j - h_j) = 2^{-j} \ .$$

Consider as the problem object the formal function $F \in \mathbb{C}[[x,y,z]]$ given by

$$F = z - y - \phi$$

and fix a value $\gamma \geq 1$. The function F is not γ-prepared (note that the 0-level $F_0 = -y - \phi$ has explicit value $0 < \gamma$ and it is not 0-final dominant). In order to γ-prepare F is enough to perform a y-Puiseux's package $\mathcal{A} \to \mathcal{A}'_1$, formed by two blow-ups, whose equations are

$$x = x_1^2(y_1+1) \ , \quad y = x_1(y_1+1) \ ,$$

where (x_1, y_1, z) are the local coordinates in $\mathcal{A}'_1$. We have $x_1 = f_1$ and $y_1 = g_1$. Moreover,

$$F = z - x_1(y_1+1) - \phi \ ,$$

thus F is γ-prepared (note that x_1^2 divides ϕ). In $\mathcal{A}'_1$ the ramification index of the dependent variable z is 1, therefore the combinatorial blow-up with center (x_1, z) is a z-Puiseux's package $\mathcal{A}'_1 \to \mathcal{A}_1$. We have local coordinates (x_1, y_1, z_1) where z_1 is given by

$$z = x_1(z_1+1) \ .$$

It follows that

$$F = x_1(z_1 + \xi_1) - x_1(y_1 + \xi_1) - \phi = x_1\left(z_1 - y_1 - \frac{\phi}{x_1}\right) \ .$$

The situation in $\mathcal{A}_1$ is similar to the one we have in $\mathcal{A}$,but now the variables have half value than the original ones. We can iterate this process by considering an infinite sequence

$$\mathcal{A} \xrightarrow{\pi_1} \mathcal{A}'_1 \xrightarrow{\tau_1} \mathcal{A}_1 \xrightarrow{\pi_2} \cdots$$

where $\pi_i : \mathcal{A}_i \to \mathcal{A}'_{i+1}$ is a y_i-Puiseux's package (in fact it is also a γ-preparation for F) and $\tau_i : \mathcal{A}'_i \to \mathcal{A}_i$ is a z_i-Puiseux's package. We have $\mathcal{A}_i = (\mathcal{O}_i, (x_i, y_i, z_i))$, where $\mathcal{O}_i = \mathbb{C}[z_i, y_i, z_i]_{(x_i,y_i,z_i)}$. Moreover, we know the values of the coordinates in any $\mathcal{A}_i$ since $x_i = f_i$, $y_i = g_i$ and $z_i = h_i$. Furhermore, in all these models the function F is γ-prepared and it is written as

$$F = x_i^{2^i-1} U_i \left(z_i - y_i - \sum_{k=1}^{\infty} c_k x_i^{2^i(k-1)+1} V_i \right) ,$$

where U_i and V_i are units of $\mathcal{O}_i = \mathbb{C}[z_s, y_s, z_s]_{(x_i,y_i,z_i)}$. Thus we have

$$\nu_{\mathcal{A}_i}(F) = \nu(x_i^{2^i-1}) = (2^i - 1)\nu(f_i) = 1 - 2^{-i} < 1 \leq \gamma ,$$

so F is not γ-final in any $\mathcal{A}_i$.

In order to avoid this accumulation phenomenon we perform a Tschirnhausen coordinate change $\mathcal{A} \to \mathcal{B}$ given by

$$\tilde{z} = z - y .$$

We have $\nu(\tilde{z}) = \nu(x)$ thus the ramification index of $\tilde{z}$ is 1. The formal function F is written in this variables as

$$F = \tilde{z} - \phi ,$$

thus it is γ-prepared. A combinatorial blow-up with center $(x, \tilde{z})$ is a $\tilde{z}$-Puiseux's package $\mathcal{B} \to \mathcal{B}_1$. In $\mathcal{B}_1$ we have local coordinates $(x, y, \tilde{z}_1)$ where $\tilde{z}_1$ is given by

$$\tilde{z}_1 = \frac{\tilde{z}}{x} - c_1$$

and its value is $\nu(\tilde{z}_1) = 1$. The function F is written now as

$$F = x \left(\tilde{z}_1 - \sum_{k=1}^{\infty} c_{k+1} x^k \right) ,$$

so it is γ-prepared. Performing $\tilde{z}$-Puiseux's packages

$$\mathcal{B} \xrightarrow{\theta_1} \mathcal{B}_1 \xrightarrow{\theta_2} \cdots$$

we obtain parameterized regular local models $\mathcal{B}_i$ with coordinates $(x, y, \tilde{z}_i)$, with $\nu(\tilde{z}_i) = 1$, in such a way that F is written as

$$F = x^i \left(\tilde{z}_i - \sum_{k=1}^{\infty} c_{k+i} x^k \right) .$$

In any of these models F is γ-prepared and we have

$$\nu_{\mathcal{B}_i}(F) = \nu(x^i) = i .$$

We see that for any index i greater than γ the function F is γ-final recessive in $\mathcal{B}_i$.

In this example we see that the formal function F has "infinite value" with respect to ν. The same happens if we consider the 1-form with formal coefficients dF. Moreover, we see that this accumulation phenomenon can also happen when dealing with convergent functions: once we have fixed a value $\gamma \geq 1$ is is enough to consider the function $F_T = z - y - \sum_{k=1}^{T} c_k x^k$ for any $T > \gamma$. Although $\nu(F_T) = T + 1$, while we perform Puiseux's packages exclusively, the explicit value of F will be strictly less than 1. Again, dF provides an example of the same phenomenon for 1-forms.

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Let us explain now how to γ-prepare a 1-form $\omega \in N_{\mathcal{A}}^{\ell+1}$ such that

$$\nu_{\mathcal{A}}(\omega \wedge d\omega) \geq 2\gamma \ .$$

Consider the decomposition in powers of the dependent variable z

$$\omega = \sum_{k=0}^{\infty} z^k \omega_k \ , \quad \omega_k = \eta_k + f_k \frac{dz}{z} \ ,$$

where $\eta_k \in N_{\mathcal{A}}^{\ell}$ and $f_k \in R_{\mathcal{A}}^{\ell}$. Note that unlike the case of functions, the levels ω_s of a 1-form $\omega \in N_{\mathcal{A}}^{\ell+1}$ do not belong to $N_{\mathcal{A}}^{\ell}$. For each level $\omega_s = \eta_s + f_s dz/z$ we associate a pair (η_s, f_s) in such a way that we can apply $T_5(\ell)$ as we will detail later.

The explicit value $\nu_{\mathcal{A}}(\omega_k)$ of a level ω_k is the minimum among $\nu_{\mathcal{A}}(\eta_k)$ and $\nu_{\mathcal{A}}(f_k)$. Given a value $\alpha \in \Gamma$ we say that ω_k is *α-final dominant* if one of the following conditions holds:

- $\nu_{\mathcal{A}}(\eta_k) < \nu_{\mathcal{A}}(f_k)$ and η_k is α-final dominant;
- $\nu_{\mathcal{A}}(f_k) < \nu_{\mathcal{A}}(\eta_k)$ and f_k is α-final dominant;
- $\nu_{\mathcal{A}}(f_k) = \nu_{\mathcal{A}}(\eta_k)$ and both f_k and η_k are α-final dominant.

The Truncated Newton Polygon $\mathcal{N}$ and the Truncated Dominant Newton Polygon Dom$\mathcal{N}$ of a 1-form are obtained in the same way we do it in the case of functions: considering the positive convex hull in $\mathbb{R}^2$ of the points $(\nu_{\mathcal{A}}(\omega_s), s)$ (all of them of $\mathcal{N}$, only the $(\gamma - s\nu(z))$-final dominant ones for Dom$\mathcal{N}$) together with $(\gamma, 0)$ and $(0, \gamma/\nu(z))$. In the same way, we say that ω is γ-prepared if

$$\mathcal{N} = \mathrm{Dom}\mathcal{N}$$

and moreover $(\nu_{\mathcal{A}}(\omega_s), s)$ is a interior point of $\mathcal{N}$ for all s such that ω_s is non-dominant.

As in the case of functions, the first step in the process of γ-preparation of a 1-form consists in performing a ℓ-nested transformation in order to get the maximum number of $(\gamma - s\nu(z))$-final dominant levels ω_s. This step do not use the hypothesis induction. Furthermore, performing a ℓ-nested transformation does not mix the levels between them, and inside each level there are no interferences between the differential part η_s and the functional part f_s.

After completing this first step we have that Dom$\mathcal{N}$ is stable, it means, Dom$\mathcal{N}$ will be the same even if we perform more ℓ-nested transformations. We have $\mathcal{N} \subset$ Dom$\mathcal{N}$ and we must to determine a transformation which make both polygons equal (the remaining condition to reach a γ-prepared situation is easy to obtain once both polygons are equal).

In order to apply $T_5(\ell)$ to the levels ω_s we need to know $\nu_{\mathcal{A}}(\eta_s \wedge d\eta_s)$. We have that $\nu_{\mathcal{A}}(\omega \wedge d\omega) \geq 2\gamma$, however, we do not know if the same happens with the 3-forms $\eta_s \wedge d\eta_s$. In order to know "how integrable" is each η_s we use the following formula:

$$\omega \wedge d\omega = \sum_{m=0}^{\infty} z^m \Big(\Theta_m + \frac{dz}{z} \Delta_m \Big)$$

where

$$\Theta_m := \sum_{i+j=m} \eta_i \wedge d\eta_j$$

and

$$\Delta_m := \sum_{i+j=m} j\eta_j \wedge \eta_i + f_i d\eta_j + \eta_i \wedge df_j \ .$$

Since $\nu_{\mathcal{A}}(\omega \wedge d\omega) \geq 2\gamma$ we have

$$\nu_{\mathcal{A}}(\Theta_m) \geq 2\gamma \quad \text{and} \quad \nu_{\mathcal{A}}(\Delta_m) \geq 2\gamma$$

for every $m \geq 0$. On the other hand, for all 1-form σ we have

$$\nu_A(d\sigma) \geq \nu_A(\sigma) \ .$$

Thanks to this observation, we have that $\nu_{\mathcal{A}}(\Theta_{2s}) \geq 2\gamma$ implies

$$\nu_{\mathcal{A}}(\eta_s \wedge d\eta_s) \geq \min \big\{ \{2\gamma\} \cup \{\nu_{\mathcal{A}}(\eta_{s-i}) + \nu_{\mathcal{A}}(\eta_{s+i})\}_{1 \leq i \leq s} \big\} \ .$$

This formula allows us to apply $T_5(\ell)$ starting with the lowest non-dominant level and continuing with the upper levels until reach the highest non-dominant level below $\gamma/\nu(z)$. Repeating this process we can approach $\mathcal{N}$ to Dom$\mathcal{N}$ until a prefixed distance as it is explained in Lemma 10. In particular we can get that all the vertices of Dom$\mathcal{N}$ are also vertices of $\mathcal{N}$. This procedure is essential but it is not enough to complete the γ-preparation.

After bringing $\mathcal{N}$ over Dom$\mathcal{N}$ we complete the γ-preparation by using "truncated proportionality" statements as the following truncated version of the De Rham-Saito Lemma ([20]):

> Let η and σ be 1-forms. If η is log-elementary and $\nu_{\mathcal{A}}(\eta \wedge \sigma) = \alpha$ then there is a function f and a 1-form $\bar{\sigma}$ with $\nu_{\mathcal{A}}(\sigma) > \alpha$ such that $\sigma = f\eta + \bar{\sigma}$.

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Once we have completed the γ-preparation process, we consider, as we did in the case of functions, the critical height χ of $\mathcal{N}$ as the main control invariant. We must to study the behavior of χ after performing a z-Puiseux's package, with rational contact function $z^d/\boldsymbol{x}^{\boldsymbol{p}}$, followed by a new γ-preparation. Unless one of the *resonant conditions* (R1) or (R2) defined in Chapter 7 is satisfied, the critical height drops. If at one moment we have $\chi = 0$, one more z-Puiseux's package followed by a γ-preparation will produce a γ-final situation.

Condition (R1) requires the ramification exponent d to be strictly greater than 1. As we will show, if this condition is satisfied it can not be satisfied again while the critical height remains stable. On the other hand, condition (R2) requires $d = 1$, but unlike (R1), it can be satisfied infinitely many times.

Therefore, it only remains to consider the case in which condition (R1) is satisfied indefinitely. In this situation, each z-Puiseux's package increases the explicit value of ω. If it becomes greater than γ, we will reach a parameterized regular local model in which ω is γ-final recessive. However, as in the case of functions, it may happen that the explicit value "accumulates". To avoid this phenomenon, we perform a Tschirnhausen change of coordinates. Such a change of variables will be determined thanks to truncated proportionality properties similar to the ones used in the γ-preparation process.

Chapter 1

Basic notions

1.1 Codimension one foliations

Let k be an algebraically closed field of characteristic zero and consider a finitely generated field extension K/k. Let $\Omega_{K/k}$ be the module of Kähler differentials. Let $d : K \to \Omega_{K/k}$ be the exterior derivative. We have that a finite subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset K$ is a transcendence basis of K/k if and only if $\Omega_{K/k}$ is a free K-module generated by $\{d\alpha_1, d\alpha_2, \ldots, d\alpha_n\}$ (see [16], Corollary 5.4). In particular we have that $\Omega_{K/k}$ is a K-vector space of dimension

$$\dim_K(\Omega_{K/k}) = \mathrm{tr.deg}(K/k) .$$

Definition 1. A *rational codimension one foliation of* K/k is a one-dimensional K-vector subspace $\mathcal{F} \subset \Omega_{K/k}$ such that for any $\omega \in \mathcal{F}$ the *integrability condition*

$$\omega \wedge d\omega = 0$$

is satisfied.

A projective model of K/k is a projective k-variety M, in the sense of scheme theory, such that $K = K(M)$ is its field of rational functions. Let n be the dimension of the variety M. We have that $n := \dim(M) = \mathrm{tr.deg}(K/k)$. Let $P \in M$ be a point and denote by $\mathcal{O}_{M,P}$ and $\mathfrak{m}_{M,P}$ its local ring and its maximal ideal respectively. Suppose that P is k-rational, it means, the residue field $\kappa_{M,P} := \mathcal{O}_{M,P}/\mathfrak{m}_{M,P}$ is isomorphic to the ground field k (so in particular P is a closed point). The point $P \in M$ is regular if $\mathcal{O}_{M,P}$ is a regular local ring of Krull dimension n. Let $\Omega_{\mathcal{O}/k}$ be the $\mathcal{O}$-module of Kähler differentials. The *Jacobian Criterion* (see [16], Theorem 7.2) states that the point P is regular if and only if $\Omega_{\mathcal{O}/k}$ is a free $\mathcal{O}$-module of rank n.

Fix a regular point $P \in M$ and denote its local ring $\mathcal{O}_{M,P}$ by $\mathcal{O}$. Since $\mathrm{Frac}(\mathcal{O}) = K$ we have an inclusion of $\mathcal{O}$-modules

$$\Omega_{\mathcal{O}/k} \hookrightarrow \Omega_{K/k} = \Omega_{\mathcal{O}/k} \otimes_{\mathcal{O}} K .$$

Given a rational codimension one foliation $\mathcal{F} \subset \Omega_{K/k}$ and a point $P \in M$ define

$$\mathcal{F}_{M,P} := \mathcal{F} \cap \Omega_{\mathcal{O}/k} .$$

We have that $\mathcal{F}_{M,P}$ is a free rank one $\mathcal{O}$-submodule of $\Omega_{\mathcal{O}/k}$. Let $\{z_1, z_2, \ldots, z_n\}$ be a regular system of parameters of its local ring $\mathcal{O}$. The z_i's are algebraically independent over k so they are a transcendence basis of K/k, thus we have

$$\Omega_{K/k} \simeq \bigoplus_{i=1}^{n} dz_i K \quad \text{and} \quad \Omega_{\mathcal{O}/k} \simeq \bigoplus_{i=1}^{n} dz_i \mathcal{O}.$$

Take an element $\omega = \sum a_i dz_i$ of $\mathcal{F}_{M,P}$. We have that ω generates $\mathcal{F}_{M,P}$ as $\mathcal{O}$-module if and only if $a_1, a_2, \ldots, a_n$ are coprime elements of $\mathcal{O}$. In fact, let $d \in \mathcal{O}$ be the greatest common divisor of the coefficients a_i. Denote $\tilde{a}_i := d^{-1} a_i$. Since $\tilde{a}_i \in \mathcal{O}$, the 1-form $\tilde{\omega} = \sum \tilde{a}_i dz_i$ is an element of $\mathcal{F}_{M,P}$. If ω is a generator of $\mathcal{F}_{M,P}$ then d has to be a unit since $\tilde{\omega} = d^{-1}\omega$. On the other hand, if $a_1, a_2, \ldots, a_n$ are not coprimes $d^{-1} \notin \mathcal{O}$, so ω is not a generator.

Proposition 1. *The following are equivalent:*

1. *$\Omega_{\mathcal{O}/k}/\mathcal{F}_{M,P}$ is a free $\mathcal{O}$-module of rank $n-1$;*
2. *there is a decomposition $\Omega_{\mathcal{O}/k} = \mathcal{F}_{M,P} \oplus \mathcal{J}$ with $\mathcal{J}$ a free $\mathcal{O}$-module of rank $n-1$;*
3. *there exists an element $\omega = \sum a_i dz_i \in \mathcal{F}_{M,P}$ with $(a_1, a_2, \ldots, a_n) = \mathcal{O}$.*

Proof. The equivalence 1) $\Leftrightarrow$ 2) is direct. For 2) $\Rightarrow$ 3) note that $(a_1, a_2, \ldots, a_n) = \mathcal{O}$ implies that the coefficients a_i are coprimes, hence ω is a generator of $\mathcal{F}_{M,P}$. Let i_0 be an index such that a_{i_0} is a unit of $\mathcal{O}$. Taking $\mathcal{J}$ as $\mathcal{J} = \oplus_{i \neq i_0} dz_i \mathcal{O}$ we obtain 2). Finally for 1) $\Rightarrow$ 3) suppose 3) is false. We have that the classes of dz_i modulo $\mathcal{F}_{M,P}$ are $\mathcal{O}$-independents, so the rank of $\Omega_{\mathcal{O}/k}/\mathcal{F}_{M,P}$ is at least n. □

Definition 2. A rational codimension one foliation $\mathcal{F}$ is *regular at a point* $P \in M$ if P is a non-singular point of M and the equivalent conditions of Proposition 1 are satisfied.

Remark 1. Given a projective model M of K the set $\mathrm{Reg}_{\mathcal{F}}(M)$ of points where $\mathcal{F}$ is regular is a non-empty open subset of M.

Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_r)$ be the first r elements of the regular system of parameters $\{z_1, z_2, \ldots, z_n\}$ and let $\boldsymbol{y} = (y_1, y_2, \ldots, y_{n-r})$ be the remaining ones. Let $\Omega_{\mathcal{O}/k}(\log \boldsymbol{x})$ be the free $\mathcal{O}$-module

$$\Omega_{\mathcal{O}/k}(\log \boldsymbol{x}) := \left(\bigoplus_{i=1}^{r} \frac{dx_i}{x_i} \mathcal{O} \right) \oplus \left(\bigoplus_{j=1}^{n-r} dy_i \mathcal{O} \right).$$

We have $\mathcal{O}$-module monomorphisms

$$\begin{array}{rcl} \Omega_{\mathcal{O}/k} & \hookrightarrow & \Omega_{\mathcal{O}/k}(\log \boldsymbol{x}) \\ \omega = \displaystyle\sum_{i=1}^{n} a_i dz_i & \mapsto & \displaystyle\sum_{i=1}^{r} x_i a_i \frac{dx_i}{x_i} + \sum_{j=r+1}^{n} a_i dy_j \end{array} \tag{1.1}$$

and

$$\begin{array}{rcl} \Omega_{\mathcal{O}/k}(\log \boldsymbol{x}) & \hookrightarrow & \Omega_{K/k} \\ \displaystyle\sum_{i=1}^{r} a_i \frac{dx_i}{x_i} + \sum_{j=r+1}^{n} a_i dy_j & \mapsto & \displaystyle\sum_{i=1}^{r} \frac{a_i}{x_i} dx_i + \sum_{j=r+1}^{n} a_i dy_j. \end{array}$$

Given a foliation $\mathcal{F} \subset \Omega_{K/k}$ and a point $P \in M$ denote

$$\mathcal{F}_{M,P}(\log \boldsymbol{x}) := \mathcal{F} \cap \Omega_{\mathcal{O}/k}(\log \boldsymbol{x}) \ .$$

We have that $\mathcal{F}_{M,P}(\log \boldsymbol{x})$ is a rank one free $\mathcal{O}$-submodule of $\Omega_{\mathcal{O}/k}(\log \boldsymbol{x})$. Take an element $\omega = \sum a_i \frac{dx_i}{x_i} + \sum a_j dy_j \in \mathcal{F}_{M,P}(\log \boldsymbol{x})$. We have that ω generates $\mathcal{F}_{M,P}(\log \boldsymbol{x})$ as $\mathcal{O}$-module if and only if $a_1, a_2, \ldots, a_n$ are coprime elements of $\mathcal{O}$.

Definition 3. Let $P \in M$ be a closed point. Let $(\boldsymbol{x}, \boldsymbol{y})$ be a regular system of parameters of $\mathcal{O}_{M,P}$. A foliation $\mathcal{F}$ is *$\boldsymbol{x}$-log-final at P* if P is a non-singular point of M and a generator of $\mathcal{F}_{M,P}(\log \boldsymbol{x})$

$$\omega = \sum_{i=1}^{r} a_i \frac{dx_1}{x_1} + \sum_{j=r+1}^{n} a_j dy_j,$$

satisfies one of the following conditions:

1. $(a_1, a_2, \ldots, a_r) = \mathcal{O}$;
2. $(a_1, a_2, \ldots, a_r) \subset \mathfrak{m}$ and in addition

$$(a_1, a_2, \ldots, a_r) \not\subset (x_1, x_2, \ldots, x_r) + \mathfrak{m}^2 \ .$$

Points satisfying the first condition are called *$\boldsymbol{x}$-log-elementary* and the ones satisfying the second condition are called *$\boldsymbol{x}$-log-canonical.*

Definition 4. A foliation $\mathcal{F}$ is *log-final at P* if there exists a regular system of parameters $(\boldsymbol{x}, \boldsymbol{y})$ of $\mathcal{O}_{M,P}$ such that $\mathcal{F}$ is $\boldsymbol{x}$-log-final at P.

Remark 2. Given a projective model M of K the set $\text{Log-Final}_{\mathcal{F}}(M)$ of points where $\mathcal{F}$ is log-final is a non-empty open subset of M.

1.2 Valuations

We collect now some classical definitions and results from valuation Theory. We omit the proofs of many assertions, which can be found in [29], Chapter 6.

Definition 5. Let K/k be a field extension. A subring $R \subset K$ is a *valuation ring of K/k* if $\text{Frac}(R) = K$, $k \subset R$ and the following property holds:

$$\forall x \in K, \quad x \notin R \Rightarrow x^{-1} \in R.$$

It follows from the definition that R is a local ring with maximal ideal

$$\mathfrak{m} = \{x \in R \mid x^{-1} \notin R\}.$$

Definition 6. Let K be an extension field of k and let Γ be an additive abelian totally ordered group. A valuation of K/k with values in Γ is a surjective mapping

$$\nu : K^* \longrightarrow \Gamma$$

such that the following conditions are satisfied:

- $\nu(xy) = \nu(x) + \nu(y)$,
- $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$,
- $\nu(\alpha) = 0$ for every $\alpha \in k^*$.

It is usual to add formally the element $+\infty$ to the group Γ with the usual arithmetic rules ($\alpha + \infty = \infty,\ \beta < \infty\ \forall\alpha, \beta \in \Gamma$) and consider the valuation $\nu : K \longrightarrow \Gamma \cup \{\infty\}$ with $\nu(0) = \infty$. We will use this convention.

Given a valuation ν of K/k the set

$$R_\nu := \{x \in K \mid \nu(x) \geq 0\}$$

is a valuation ring of K/k with maximal ideal

$$\mathfrak{m}_\nu := \{x \in K \mid \nu(x) > 0\}.$$

The ring R_ν is *the valuation ring of* ν. Its quotient field $\kappa_\nu := R_\nu/\mathfrak{m}_\nu$ is *the residue field of* ν.

Conversely, given a valuation ring R of K/k we can construct a valuation ν_R of K/k such that $R = R_{\nu_R}$. Since R is a subring of K, its invertible elements form a subgroup $R^* = R \setminus \mathfrak{m}$ of the multiplicative group K^*. Let Γ be the quotient group K^*/R^*. It is an abelian totally ordered group whose order relation is given by the divisibility in R:

$$xR^* \leq yR^* \Leftrightarrow x \text{ divides } y \text{ in } R.$$

This is clearly an order relation on Γ. Since R is a valuation ring we have that it is a total order:

$$xR^* \not\leq yR^* \Rightarrow x \text{ does not divide } y \text{ in } R \Rightarrow \frac{y}{x} \notin R$$

$$\Rightarrow \frac{x}{y} \in R \Rightarrow y \text{ divides } x \text{ in } R \Rightarrow yR^* \leq xR^*.$$

The valuation is the natural group homomorphism $\nu : K^* \to \Gamma = K^*/R^*$. The positive part Γ_+ is the image of the maximal ideal $\mathfrak{m}$.

A totally ordered group G is *archimedean* if it satisfies the *archimedean property*:

$$\forall x, y \in G_{>0},\ \exists n \in \mathbb{N} \mid y \leq x.$$

It is well-known that a totally ordered group is archimedean if and only if it is isomorphic as totally ordered group to some subgroup of $(\mathbb{R}, +)$.

Definition 7. A valuation ν of K/k is *archimedean* if its value group Γ_ν is archimedean.

The *rank of* ν is defined by

$$\mathrm{rk}(\nu) := \dim_{Krull} R_\nu.$$

This number coincides with the rank of the ordered group Γ. We have that a valuation ν is archimedean if and only if $\mathrm{rk}(\nu) = 1$.

Given a valuation ν of K/k the residue field $\kappa_\nu := R_\nu/\mathfrak{m}_\nu$ is the *residue field of* ν. It follows from the third property of the definition of valuation of K/k that κ_ν is an extension field of k. We define the *dimension of* ν by

$$\dim(\nu) := tr.deg(\kappa_\nu/k).$$

Definition 8. A valuation ν of K/k is *k-rational* if $\kappa_\nu \simeq k$.

Remark 3. Note that in the case of an algebraically closed ground field k a valuation ν of K/k is k-rational if and only if $\dim(\nu) = 0$.

If ν is a k-rational valuation of K/k, we have that for each $\phi \in K$ with $\nu(\phi) \geq 0$ there exists a unique $\lambda \in k$ such that $\nu(\phi - \lambda) > 0$. The existence of such a $\lambda \in k$ follows from the fact that $\kappa_\nu \simeq k$. Suppose that there are $\lambda_1, \lambda_2 \in k$ such that $\nu(\phi - \lambda_i) > 0$ for $i = 1, 2$. It follows that $\nu(\lambda_1 - \lambda_2) = \nu((\phi - \lambda_2) - (\phi - \lambda_1)) > 0$ hence $\lambda_1 = \lambda_2$.

The largest number of elements of K with $\mathbb{Z}$-independent values is the *rational rank of* ν

$$\mathrm{rat.\,rk}(\nu) := \dim_{\mathbb{Q}}(\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}) \ .$$

By the Abhyankar's Inequality we have that

$$0 \leq \mathrm{rk}(\nu) \leq \mathrm{rat.\,rk}(\nu) \leq \mathrm{tr.\,deg}(K/k) = n \ .$$

Let (A, m) and (B, n) be two local rings. We say that A is dominated by B if $A \subset B$ and $m = A \cap n$. The relation of domination is denoted by $A \preceq B$.

Definition 9. Let X be an algebraic variety over k whose function field is K. A point $P \in X$ is called *the center of ν at X* if $\mathcal{O}_{X,P} \preceq R_\nu$.

We will work with projective models of a fixed function field K, i. e., algebraic projective varieties with function field K. The following proposition, whose proof can be found in [19], guarantees the existence and uniqueness of the center of any valuation of K/k in such models.

Proposition 2. *If X is a complete variety over a field k, any valuation of L/k, where $L/K(X)$ is an extension of the function field $K(X)$ of X, has a unique center on X.*

If P is a point of a variety X, its residue field is by definition $\kappa_{X,P} := \mathcal{O}_{X,P}/\mathfrak{m}_{X,P}$, which is an extension field of k. If we have $\kappa_P = k$ we say that P is a k-rational point. If P is the center of ν at X there is a tower of fields

$$k \subset \kappa_P \subset \kappa_\nu.$$

In particular we have $\dim(P) \leq \dim(\nu)$, where $\dim(P) := tr.deg(\kappa_P/k)$. Let us note that if ν is k-rational then the three fields are the same and the center of ν in each projective model has dimension 0 and it is a k-rational point.

Let $f : X' \to X$ a birational morphism. If P' and P are the centers of ν at X' and X respectively, we have $f(P') = P$ and f induces a domination of local rings $\mathcal{O}_{X,P} \preceq \mathcal{O}_{X',P'}$. As a consequence we obtain $\dim(P) \leq \dim(P')$. The proof of the next statement can also be found in [19].

Proposition 3. *Given a projective model X of K, there is a birational morphism $\pi : X' \to X$ with $\dim P' = \dim \nu$, where P' is the center of ν in X'.*

Given a valuation ν of K/k, the Local Uniformization Problem consists in determine a projective model M of K such that the center of ν in M is a regular point. This problem for varieties over a ground field of characteristic zero was stated and solved by Zariski (see [26]). In this work, instead of regularity at the center of the valuation, we require that a given rational codimension one foliation is log-final at the center of the valuation. The precise statement we prove in this work is the following refinement of Theorem I:

Theorem 1. *Let k be a field of characteristic zero and let K/k be a finitely generated field extension. Let $\mathcal{F}$ be a rational codimension one foliation of K/k. Given a projective model M of K/k and a k-rational archimedean valuation ν of K/k, there is a finite composition of blow-ups with codimension two centers*

$$\tilde{M} \to M$$

such that $\mathcal{F}$ is log-final at the center of ν in $\tilde{M}$.

Given a function field K/k, we can invoke Hironaka's Resolution of Singularities [14, 15] or Zariski's Local Uniformization [26] in order to obtain a projective model of K/k regular at the center of the valuation ν. In that situation we have that Theorem 1 implies Theorem I.

Chapter 2

Transformations adapted to a valuation

2.1 Parameterized regular local models

Let K/k be a finitely generated field extension and let ν be a k-rational archimedean valuation of K/k of rational rank r.

Definition 10. A *parameterized regular local model* for $K/k, \nu$ is a pair $\mathcal{A} = \big(\mathcal{O}; (\boldsymbol{x}, \boldsymbol{y})\big)$ such that

- $\mathcal{O} \subset K$ is the regular local ring of the center of ν in some projective model of K;
- $(\boldsymbol{x}, \boldsymbol{y}) = (x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_{n-r})$ is a regular system of parameters of $\mathcal{O}$ such that $\{\nu(x_1), \nu(x_2), \ldots, \nu(x_r)\} \subset \Gamma$ is a basis of $\Gamma \otimes \mathbb{Q}$. We call $\boldsymbol{x}$ the *independent variables* and $\boldsymbol{y}$ the *dependent variables.*

The following proposition guarantees the existence of parameterized regular local models.

Proposition 4. *Given a projective model M_0 of K, there is a morphism $M \longrightarrow M_0$ which is the composition of blow-ups with non-singular centers, such that the center P of ν at M provides a local ring $\mathcal{O} = \mathcal{O}_{M,P}$ for a parameterized regular local model $\mathcal{A}$ for $K/k, \nu$.*

Proof. By Zariski's Local Uniformization [26] we get a projective model M' of K non-singular at the center P' of ν, jointly with a birational morphism $M' \to M_0$ that is the composition of a finite sequence of blow-ups with non-singular centers.

Take r elements $g_1, g_2, \ldots, g_r \in K$ such that $\nu(g_1), \nu(g_2), \ldots, \nu(g_r)$ are $\mathbb{Z}$-linearly independent. Since $K = \mathrm{Frac}(\mathcal{O}_{M',P'})$, multiplying by the common denominator if necessary we obtain r elements $f_1, f_2, \ldots, f_r \in \mathcal{O}_{M',P'}$ with independent values. Another application of Zariski's Local Uniformization gives a birational morphism $M \to M'$, that is also a composition of a finite sequence of blow-ups with non-singular centers, such that each f_i is a monomial (times a unit) in a suitable regular system of parameters of $\mathcal{O}_{M,P}$, where P is the center of ν in M. Let $\boldsymbol{z} = (z_1, z_2, \dots, z_n)$ be such a regular system of parameters. We have

$$f_i = U_i \boldsymbol{z}^{\boldsymbol{m_i}} \quad , U_i \in \mathcal{O}_{M,P} \setminus \mathfrak{m}_{M,P} \text{ for } i = 1, 2, \dots, r,$$

where $\boldsymbol{m_i} \in \mathbb{Z}^n_{\geq 0}$. In terms of values, we have $\nu(f_i) = \sum m_{ij}\nu(z_j)$. This implies that there are r variables among the z_j's whose values are $\mathbb{Z}$-linearly independent. □

2.2 Transformations of parameterized regular local models

Parameterized regular local models are the ambient spaces in which we will work. A transformation between parameterized regular local models is formed by an inclusion of regular local rings and a specific selection of a regular system of parameters of the new ring. We consider two elementary operations: blow-ups and change of coordinates. Certain composition of these operations, called nested transformations, are the transformations between parameterized regular local models which we allow. Given such a transformation

$$\pi : \mathcal{A} \longrightarrow \mathcal{A}' \ ,$$

we denote with the same symbol the corresponding inclusion of local rings

$$\pi : \mathcal{O} \longrightarrow \mathcal{O}' \ ,$$

and also the induced $\mathcal{O}$-module homomorphism

$$\pi : \Omega_{\mathcal{O}/k}(\log \boldsymbol{x}) \longrightarrow \Omega_{\mathcal{O}'/k}(\log \boldsymbol{x}') \ .$$

2.2.1 Blowing-up parameterized regular local models

Let $\mathcal{A} = \big(\mathcal{O}; (\boldsymbol{x}, \boldsymbol{y})\big)$ be a parametrized regular local model for K, ν. As center of blow-up we use only the ideals $I_{ij}, I_i^j \subset \mathcal{O}$ defined by

$$\begin{aligned} I_{ij} &:= (x_i, x_j)\mathcal{O} \quad \text{for} \quad 1 \leq i < j \leq r \ ; \\ I_i^j &:= (x_i, y_j)\mathcal{O} \quad \text{for} \quad 1 \leq i \leq r, \ 1 \leq j \leq n - r \ . \end{aligned}$$

The blow-up of $\mathcal{A}$ at I_{ij} is

$$\theta_{ij}(\mathcal{A}) : \mathcal{A} \longrightarrow \mathcal{A}' \ ,$$

where $\mathcal{A}' = \big(\mathcal{O}'; (\boldsymbol{x}', \boldsymbol{y}')\big)$ is defined by:

- if $1 \leq i < j \leq r$ and $\nu(x_i) < \nu(x_j)$: put $x'_j := x_j/x_i$, $x'_k = x_k$ for $k \neq j$ and $\boldsymbol{y}' := \boldsymbol{y}$;
- if $1 \leq i < j \leq r$ and $\nu(x_i) > \nu(x_j)$: put $x'_i := x_i/x_j$, $x'_k = x_k$ for $k \neq i$ and $\boldsymbol{y}' := \boldsymbol{y}$;

and

$$\mathcal{O}' = \mathcal{O}[\boldsymbol{x}', \boldsymbol{y}']_{(\boldsymbol{x}', \boldsymbol{y}')} \ .$$

The blow-up of $\mathcal{A}$ at I_i^j is

$$\theta_i^j(\mathcal{A}) : \mathcal{A} \longrightarrow \mathcal{A}' \ ,$$

where $\mathcal{A}' = \big(\mathcal{O}'; (\boldsymbol{x}', \boldsymbol{y}')\big)$ is defined by:

- if $\nu(x_i) < \nu(y_j)$: put $y'_j := y_j/x_i$, $y'_j = y_j$ for $j \neq l$ and $\boldsymbol{x}' := \boldsymbol{x}$;
- if $\nu(x_i) > \nu(y_j)$: put $x'_i := x_i/y_j$, $x'_i = x_i$ for $i \neq k$ and $\boldsymbol{y}' := \boldsymbol{y}$;
- if $\nu(x_i) = \nu(y_j)$: since $\kappa_\nu = k$, there is $\lambda \in k$ with $\nu(y_j/x_i - \lambda) > 0$. Put $y'_j = y_j/x_i - \lambda$, $y'_k = y_k$ for $k \neq j$ and $\boldsymbol{x}' := \boldsymbol{x}$;

and

$$\mathcal{O}' = \mathcal{O}[\boldsymbol{x}', \boldsymbol{y}']_{(\boldsymbol{x}', \boldsymbol{y}')} \ .$$

The first four cases above are called *combinatorial blow-ups* and the fifth is a *blow-up with translation.*

Remark 4. As we said $\mathcal{O}$ is the local ring of a point P, the center of ν in some projective model M. The ring $\mathcal{O}'$ is just the local ring of the center of ν in the variety obtained after blowing-up M at the subvariety defined locally at P by I_{ij}. We have that

$$\mathcal{O} \preceq \mathcal{O}' \preceq R_\nu \ .$$

2.2.2 Ordered change of coordinates

A change of coordinates does not modify the local ring, it just changes the selection of regular parameters. Let $\mathcal{A} = \big(\mathcal{O}; (\boldsymbol{x}, \boldsymbol{y})\big)$ be a parameterized regular local model and let y_ℓ be a dependent variable.

Definition 11. An *ordered change of the ℓ-th coordinate* is a transformation of parametrized regular local models

$$T : \mathcal{A} \longrightarrow \tilde{\mathcal{A}}$$

where $\tilde{\mathcal{A}} = \big(\tilde{\mathcal{O}}; (\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})\big)$ is given by

- $\tilde{\mathcal{O}} := \mathcal{O}$;
- $\tilde{x}_i := x_i$ for $1 \leq i \leq r$;
- $\tilde{y}_j := y_j$ for $1 \leq j \leq n-r$, $j \neq \ell$;
- $\tilde{y}_\ell := y_\ell + \psi$ where $\psi \in \mathfrak{m} \cap k[\boldsymbol{x}, y_1, y_2, \ldots, y_{l-1}]$ is a polynomial such that if we write
$$\psi = \sum_I \boldsymbol{x}^I \psi_I(y_1, y_2, \ldots, y_{\ell-1})$$
we have
$$\nu(\boldsymbol{x}^I) < \nu(y_\ell) \Rightarrow \psi_I(y_1, y_2, \ldots, y_{\ell-1}) \equiv 0 \ .$$

Note that we have

$$\nu(\tilde{y}_\ell) \geq \nu(y_\ell) \ .$$

Taking differentials in the equations of the coordinate change we obtain explicit equations for the $\mathcal{O}$-homomorphism between the modules of differentials

$$\begin{array}{rcl}
\Omega_{\mathcal{O}/k}(\log \boldsymbol{x}) & \longrightarrow & \Omega_{\tilde{\mathcal{O}}/k}(\log \tilde{\boldsymbol{x}}) \\
\dfrac{dx_i}{x_i} & \longmapsto & \dfrac{d\tilde{x}_i}{\tilde{x}_i} \ , \quad i = 1, \ldots, r \ ; \\
dy_j & \longmapsto & d\tilde{y}_j \ , \quad 1 \leq j \leq n-r \ , \ j \neq \ell \ ; \\
dy_\ell & \longmapsto & d\tilde{y}_\ell + \displaystyle\sum_{i=1}^{r} x_i \frac{\partial \psi}{\partial x_i} \frac{dx_i}{x_i} + \sum_{j=1}^{\ell-1} \frac{\partial \psi}{\partial y_j} dy_j \ .
\end{array}$$

2.2.3 Puiseux's packages

Let $\mathcal{A} = \big(\mathcal{O}; (\boldsymbol{x}, \boldsymbol{y})\big)$ be a parameterized regular local model for $K/k, \nu$. Given a dependent variable y_ℓ there is a relation

$$d\nu(y_\ell) = p_1\nu(x_1) + \cdots + p_r\nu(x_r). \tag{2.1}$$

Requiring $d > 0$ and $\gcd(d, p_1, \ldots, p_r) = 1$ the integers of the above expression are uniquely determined. Denoting by $\boldsymbol{p} = (p_1, \ldots, p_r) \in \mathbb{Z}^r \setminus \{\boldsymbol{0}\}$, Equation (2.1) is equivalent to

$$\nu(\frac{y_\ell^d}{\boldsymbol{x}^{\boldsymbol{p}}}) = 0. \tag{2.2}$$

The rational function $\phi_\ell := y_\ell^d/\boldsymbol{x}^{\boldsymbol{p}}$ is called the *l-th contact rational function* and $d = d(\ell; \mathcal{A})$ is the *l-ramification index*.

Remark 5. Let ϕ_ℓ be the l-th contact rational function and perform a blow-up $\mathcal{A} \to \mathcal{B}$. If π is combinatorial then the new l-th contact rational function is the strict transform of ϕ_ℓ.

Recall that there exists a unique constant $\xi \in k^*$ such that $\nu(\phi_\ell - \xi) > 0$.

Definition 12. A *ℓ-Puiseux's package* is a finite sequence of blow-ups

$$\mathcal{A} \xrightarrow{\pi_0} \mathcal{A}_1 \xrightarrow{\pi_1} \cdots \xrightarrow{\pi_N} \mathcal{A}'$$

where

- for $0 \le t \le N-1$, π_t is a combinatorial blow-up $\pi_t = \theta_{ij}(\mathcal{A}_t)$ with $1 \le i < j \le r$, or $\pi_t = \theta_i^\ell(\mathcal{A}_t)$ with $1 \le i \le r$, ;
- $\pi_N = \theta_i^l(\mathcal{A}_N)$ is a blow-up with translation.

In the special case in which the ramification index is $d(\ell; \mathcal{A}_t) = 1$ and $\nu(y_{t,\ell}) < \nu(x_{t,i})$, where $\mathcal{A}_t = (\mathcal{O}_t, (\boldsymbol{x}_t, \boldsymbol{y}_t))$, we require the combinatorial blow-up π_t not to be $\theta_i^\ell(\mathcal{A}_t)$.

The last condition in the definition is not necessary. We put it because it makes easier some calculations which will appear later in the text.

Remark 6. Let $\pi_N = \theta_i^\ell(\mathcal{A}_N) : \mathcal{A}_N \to \mathcal{A}'$ be the last blow-up of a ℓ-Puiseux's package. The l-th contact rational function in $\mathcal{A}_N$ has to be necessarily $\phi_\ell = y_\ell/x_i$ and then after the transformation we obtain $y'_\ell = \phi_\ell - \xi$.

Note that a ℓ-Puiseux's package gives a local uniformization of the hypersurface $\boldsymbol{x}^{\boldsymbol{q}} y_\ell^d - \xi \boldsymbol{x}^{\boldsymbol{t}} = 0$. We will prove the existence of Puiseux's packages in Proposition 5.

Equations of a Puiseux's package

We collect here some specific calculations about Puiseux's packages for further references. Let $\mathcal{A}' = \big(\mathcal{O}', (\boldsymbol{x}', \boldsymbol{y}')\big)$ be the parameterized regular local model obtained from $\mathcal{A} = \big(\mathcal{O}; (\boldsymbol{x}, \boldsymbol{y})\big)$ by means of a ℓ-Puiseux's package $\pi : \mathcal{A} \to \mathcal{A}'$. We have

$$\begin{array}{rcl} y'_\ell &=& \phi_\ell - \xi, \\ x_i &=& \boldsymbol{x}'^{\boldsymbol{\alpha}_i}(y'_\ell + \xi)^{\beta_i}, \\ y_\ell &=& \boldsymbol{x}'^{\boldsymbol{\alpha}_0}(y'_\ell + \xi)^{\beta_0}, \\ y_j &=& y'_j \text{ if } j \neq \ell, \end{array} \tag{2.3}$$

where $\boldsymbol{\alpha_0}, \boldsymbol{\alpha_i} \in \mathbb{Z}^r_{\geq 0}$ and $\beta_0, \beta_i \in \mathbb{Z}_{\geq 0}$. The relation (2.2) gives

$$p_1\boldsymbol{\alpha_1} + \cdots + p_r\boldsymbol{\alpha_r} - d\boldsymbol{\alpha_0} = 0, \tag{2.4}$$

$$p_1\beta_1 + \cdots + p_r\beta_r + 1 = d\beta_0. \tag{2.5}$$

It follows from the construction that the matrices $\check{H}_\pi$ and H_π defined by

$$\check{H}_\pi := \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1r} \\ \vdots & \ddots & \vdots \\ \alpha_{r1} & \cdots & \alpha_{rr} \end{pmatrix}, \tag{2.6}$$

and

$$H_\pi := \left(\begin{array}{ccc|c} & & & \beta_1 \\ & \check{H} & & \vdots \\ & & & \beta_r \\ \hline \alpha_{01} & \cdots & \alpha_{0r} & \beta_0 \end{array}\right) \tag{2.7}$$

are invertible with non-negative integers coefficients. Using matrix notation Equality (2.4) can be written as

$$\boldsymbol{p}\check{H}_\pi = -d\boldsymbol{\alpha}_0 \ . \tag{2.8}$$

Taking differentials in the expressions (2.3) we obtain

$$\begin{pmatrix} \frac{dx_1}{x_1} \\ \vdots \\ \frac{dx_r}{x_r} \\ \frac{dy_\ell}{y_\ell} \end{pmatrix} = H \begin{pmatrix} \frac{dx'_1}{x'_1} \\ \vdots \\ \frac{dx'_r}{x'_r} \\ \frac{d\phi_\ell}{\phi_\ell} \end{pmatrix} \tag{2.9}$$

where $\frac{d\phi_\ell}{\phi_\ell} = y'_\ell \phi_\ell^{-1} \frac{dy'_\ell}{y'_\ell}$. The equality (2.9) provides explicit equations for the $\mathcal{O}$-homomorphism between the modules of differentials

$$\begin{array}{rcl} \Omega_{\mathcal{O}/k}(\log \boldsymbol{x}) & \longrightarrow & \Omega_{\mathcal{O}'/k}(\log \boldsymbol{x}') \\ \dfrac{dx_i}{x_i} & \longmapsto & \displaystyle\sum_{k=1}^r \alpha_{ik}\frac{dx'_k}{x'_k} + \phi_\ell^{-1}\beta_i dy'_\ell \ , \quad i = 1, \ldots, r \ ; \\ dy_j & \longmapsto & dy'_j \ , \quad 1 \leq j \leq n-r \ , \ j \neq \ell \ ; \\ dy_\ell & \longmapsto & \boldsymbol{x}'^{\boldsymbol{\alpha}_0}\phi_\ell^{\beta_0}\left(\displaystyle\sum_{k=1}^r \alpha_{0k}\frac{dx'_k}{x'_k} + \phi_\ell^{-1}\beta_0 dy'_\ell\right) \ . \end{array}$$

.

Remark 7. Let $\boldsymbol{x}^{\boldsymbol{q}_0} y_\ell^{e_0}$ be a monomial with integer exponents and let $\gamma_0 \in \Gamma$ be its value. We have that all the monomials in the variables $\boldsymbol{x}$ and y_ℓ with value γ_0 are those of the form

$$\boldsymbol{x}^{\boldsymbol{q}_0} y_\ell^{e_0} \phi_\ell^t = \boldsymbol{x}^{\boldsymbol{q}_0 - t\boldsymbol{p}} y_\ell^{e_0 + td}.$$

After a ℓ-Puiseux's package such a monomial becomes a polynomial with the same value

$$\boldsymbol{x}^{\boldsymbol{q}_0} y_\ell^{e_0} \phi_\ell^t = \boldsymbol{x}'^{\boldsymbol{q}'_0}(y'_\ell + \xi)^t$$

where the exponent $\boldsymbol{q}'_0$ is determined by (2.3) and it does not depend on the parameter t.

Puiseux's packages without ramification

One notable case is when $d=1$ and $\boldsymbol{p}\in\mathbb{Z}^r_{\geq 0}$. In this situation we always can determine a ℓ-Puiseux's package

$$\mathcal{A}_0 \xrightarrow{\theta_{i_0\ell}} \mathcal{A}_1 \xrightarrow{\theta_{i_1\ell}} \cdots \xrightarrow{\theta_{i_{N-1}\ell}} \mathcal{A}_N$$

such that $\check{H}_\pi = I_r$ and

$$H_\pi = \left(\begin{array}{ccc|c} & & & 0 \\ & I_r & & \vdots \\ & & & 0 \\ \hline p_1 & \cdots & p_r & 1 \end{array}\right).$$

If we have $d=1$ but $\boldsymbol{p}\notin\mathbb{Z}^r_{\geq 0}$, as we will see in Lemma 3, we can determine a finite composition of blow-ups with centers of the kind I_{ij} to reach the previous case. If C is the $r\times r$ matrix related to that transformation, and $\boldsymbol{p'}\in\mathbb{Z}^r_{\geq 0}$ is the new exponent (in fact $\boldsymbol{p'}=\boldsymbol{p}C$), we have that the matrix of the complete Puiseux's package is

$$H_\pi = \left(\begin{array}{ccc|c} & & & 0 \\ & C & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & 1 \end{array}\right) \times \left(\begin{array}{ccc|c} & & & 0 \\ & I_r & & \vdots \\ & & & 0 \\ \hline p'_1 & \cdots & p'_r & 1 \end{array}\right) = \left(\begin{array}{ccc|c} & & & 0 \\ & C & & \vdots \\ & & & 0 \\ \hline p'_1 & \cdots & p'_r & 1 \end{array}\right)$$

Remark 8. In the case $d=1$ and $\boldsymbol{p}\in\mathbb{Z}^r_{\geq 0}$ there is a coordinate change related to the Puiseux's package. Let $T_0:\mathcal{A}_0\to\tilde{\mathcal{A}}_0$ be an ordered change of the ℓ-th variable given by $\tilde{y}_\ell := y_\ell - \xi\boldsymbol{x}^{\boldsymbol{p}}$. We have the following commutative diagram

$$\begin{tikzcd}
\mathcal{A}_0 \arrow[r, "\theta_{i_0\ell}"] \arrow[d, "T_0"] & \mathcal{A}_1 \arrow[r, "\theta_{i_1\ell}"] \arrow[d, "T_1"] & \cdots \arrow[r, "\theta_{i_{N-2}\ell}"] & \mathcal{A}_{N-1} \arrow[r, "\theta_{i_{N-1}\ell}"] \arrow[d, "T_{N-1}"] & \mathcal{A}_N \arrow[d, "T_N=\mathrm{id}"] \\
\tilde{\mathcal{A}}_0 \arrow[r, "\theta_{i_0\ell}"] & \tilde{\mathcal{A}}_1 \arrow[r, "\theta_{i_1\ell}"] & \cdots \arrow[r, "\theta_{i_{N-2}\ell}"] & \tilde{\mathcal{A}}_{N-1} \arrow[r, "\theta_{i_{N-1}\ell}"] & \tilde{\mathcal{A}}_N
\end{tikzcd}$$

where the upper horizontal row is a ℓ-th Puiseux's package, the vertical arrows $T_s:\mathcal{A}_s\to\tilde{\mathcal{A}}_s$ for $s=0,1,\ldots,N-1$ are ordered changes of the ℓ-th coordinate and the last vertical arrow is the identity map. Moreover, all the horizontal arrows are combinatorial blow-ups except $\theta_{i_{N-1}\ell}:\mathcal{A}_{N-1}\to\mathcal{A}_N$ which is a blow-up with translation.

2.2.4 Nested transformations

The transformations of parameterized regular models that we have introduced *respect* the relative ordering in the dependent variables. This fact is a key feature of our induction treatment.

Definition 13. A 0-*nested transformation* is a composition of transformations of parameterized regular local models

$$\mathcal{A}=\mathcal{A}_0 \xrightarrow{\tau_0} \mathcal{A}_1 \xrightarrow{\tau_1} \cdots \xrightarrow{\tau_{t-1}} \mathcal{A}_t = \mathcal{A}'$$

where each

$$\mathcal{A}_k \xrightarrow{\tau_k} \mathcal{A}_{k+1}$$

is a combinatorial blow-up $\tau_k = \theta_{i_k j_k}(\mathcal{A}_k)$ with $1 \leq i_k < j_k \leq r$.

A *ℓ-nested transformation* is a composition of transformations of parameterized regular local models

$$\mathcal{A} = \mathcal{A}_0 \xrightarrow{\tau_0} \mathcal{A}_1 \xrightarrow{\tau_1} \cdots \xrightarrow{\tau_{t-1}} \mathcal{A}_t = \mathcal{A}'$$

where each

$$\mathcal{A}_k \xrightarrow{\tau_k} \mathcal{A}_{k+1}$$

is a $(l-1)$-nested transformation, a ℓ-Puiseux's package or an ordered change of the l-th coordinate.

Remark 9. Note that a ℓ-nested transformations is also a l'-nested transformation for $l' > l$, and in particular a $(n-r)$-nested transformation. We will refer to $(n-r)$-nested transformations simply by *nested transformations*.

2.3 Statements in terms of parameterized regular local models

Let K/k be a finitely generated field extension and let ν be a k-rational valuation. Let $\mathcal{A} = \big(\mathcal{O}; (\boldsymbol{x}, \boldsymbol{y})\big)$ be a parameterized regular local model of K, ν. Consider a codimension one rational foliation $\mathcal{F}$ of K/k. Denote by $\mathcal{F}_{\mathcal{A}}$ the submodule of $\Omega_{\mathcal{O}/k}(\log \boldsymbol{x})$ given by

$$\mathcal{F}_{\mathcal{A}} := \mathcal{F} \cap \Omega_{\mathcal{O}/k}(\log \boldsymbol{x}) \ .$$

Definition 14. A codimension one rational foliation $\mathcal{F}$ is *$\mathcal{A}$-final* if $\mathcal{F}_{\mathcal{A}}$ is $\boldsymbol{x}$-log-final,

Theorem 2. *Let k be a field of characteristic zero and let K/k be a finitely generated field extension. Let $\mathcal{F}$ be a rational codimension one foliation of K/k. Given a k-rational archimedean valuation ν of K/k and a parameterized regular local model $\mathcal{A}$ of $K/k, \nu$, there is a nested transformation*

$$\mathcal{A} \longrightarrow \mathcal{B}$$

such that $\mathcal{F}$ is $\mathcal{B}$-final.

This statement is a refinement of Theorem II, in which we specify the kind of transformations we allow. Thanks to Proposition 4 we have that Theorem 2 implies Theorem 1.

2.4 Formal completion

Although the result we want to show is about "convergent" foliations, in order to prove it we have to consider formal functions and 1-forms with formal coefficients.

Let $\mathcal{A} = \big(\mathcal{O}; (\boldsymbol{x}, \boldsymbol{y})\big)$ be a parameterized regular local model for $K/k, \nu$. Let $\hat{\mathcal{O}}$ be the $\mathfrak{m}$-adic completion of $\mathcal{O}$. By Cohen's Structure Theorem we know that

$$\hat{\mathcal{O}} \cong \kappa[[\boldsymbol{x}, \boldsymbol{y}]]$$

where $\kappa := \mathcal{O}/\mathfrak{m}$ is the residue field of the center of the valuation. Since we are dealing with k-rational valuations we have that $\kappa = k$ so in our case

$$\hat{\mathcal{O}} = k[[\boldsymbol{x}, \boldsymbol{y}]] \ .$$

Let $R^{\ell}_{\mathcal{A}}$ be the subrings of $\hat{\mathcal{O}}$ defined by

$$R^0_{\mathcal{A}} := \kappa[[\boldsymbol{x}]] \quad \text{and} \quad R^{\ell}_{\mathcal{A}} := \kappa[[\boldsymbol{x}, y_1, y_2, \ldots, y_\ell]]$$

for $1 \leq \ell \leq n - r$. All of them are local rings with maximal ideal $R^{\ell}_{\mathcal{A}} \cap \hat{\mathfrak{m}}$. We have

$$R^0_{\mathcal{A}} \subset R^1_{\mathcal{A}} \subset \cdots \subset R^{n-r}_{\mathcal{A}} = \hat{\mathcal{O}},$$

where each inclusion is in fact a relation of domination of local rings.

Consider now a ℓ-nested transformation $\pi : \mathcal{A} \longrightarrow \mathcal{A}'$. Tensoring the inclusion of local rings $\pi : \mathcal{O} \to \mathcal{O}'$ by $\hat{\mathcal{O}}$ we obtain an inclusion of complete local rings

$$\pi : \hat{\mathcal{O}} \hookrightarrow \hat{\mathcal{O}}'$$

which we denote with the same symbol π. Such an inclusion is compatible with the decomposition in subrings of $\hat{\mathcal{O}}$ in the following way:

$$\pi(R^j_{\mathcal{A}}) \subset R^{\ell}_{\mathcal{A}'} \quad \text{for} \quad 0 \leq j \leq \ell \ ,$$

and

$$\pi(R^k_{\mathcal{A}}) \subset R^k_{\mathcal{A}'} \quad \text{for} \quad \ell + 1 \leq k \leq n - r \ .$$

In fact, we have that

$$\pi_{|R^{\ell}_{\mathcal{A}}} : R^{\ell}_{\mathcal{A}} \to R^{\ell}_{\mathcal{A}'}$$

is an injective $R^{\ell}_{\mathcal{A}}$-homomorphism.

We develop now a similar construction for the modules of differentials. Let $\hat{\Omega}_{\mathcal{O}/k}$ be the free $\hat{\mathcal{O}}$-module generated by

$$\{dx_1, dx_2, \ldots, dx_r, dy_1, dy_2, \ldots, dy_{n-r}\} \ .$$

We have that

$$\hat{\Omega}_{\mathcal{O}/k} \simeq \Omega_{\mathcal{O}/k} \otimes_{\mathcal{O}} \hat{\mathcal{O}} \ .$$

Let $\hat{\Omega}_{\mathcal{O}/k}(\log \boldsymbol{x})$ be the free $\hat{\mathcal{O}}$-module generated by

$$\{\frac{dx_1}{x_1}, \frac{dx_2}{x_2}, \ldots, \frac{dx_r}{x_r}, dy_1, dy_2, \ldots, dy_{n-r}\} \ .$$

We have that

$$\hat{\Omega}_{\mathcal{O}/k}(\log \boldsymbol{x}) \simeq \Omega_{\mathcal{O}/k}(\log \boldsymbol{x}) \otimes_{\mathcal{O}} \hat{\mathcal{O}} \ .$$

Tensoring the inclusion homomorphism (1.1) by $\hat{\mathcal{O}}$ we obtain an injective $\hat{\mathcal{O}}$-module homomorphism

$$\begin{array}{rcl} \hat{\Omega}_{\mathcal{O}/k} & \to & \hat{\Omega}_{\mathcal{O}/k}(\log \boldsymbol{x}) \\ \displaystyle\sum_{i=1}^{r} a_i dx_i + \sum_{j=r+1}^{n} a_i dy_j & \mapsto & \displaystyle\sum_{i=1}^{r} x_i a_i \frac{dx_i}{x_i} + \sum_{j=r+1}^{n} a_i dy_j. \end{array}$$

For each index index ℓ, $1 \leq \ell \leq n-r$, let $N_{\mathcal{A}}^{\ell}$ be the free $R_{\mathcal{A}}^{\ell}$-module generated by

$$\{\frac{dx_1}{x_1}, \frac{dx_2}{x_2}, \ldots, \frac{dx_r}{x_r}, dy_1, dy_2, \ldots, dy_\ell\} \ .$$

Note that all these modules are subsets of $\hat{\Omega}_{\mathcal{O}/k}(\log \boldsymbol{x})$, and we have

$$N_{\mathcal{A}}^{0} \subset N_{\mathcal{A}}^{1} \subset \cdots \subset N_{\mathcal{A}}^{n-r} \cong \hat{\Omega}_{\mathcal{O}/k}(\log \boldsymbol{x}) \ ,$$

where each inclusion is just an inclusion of subsets (not a module monomorphism).

Consider now a ℓ-nested transformation $\pi : \mathcal{A} \longrightarrow \mathcal{A}'$. The inclusion of $\hat{\mathcal{O}}$-modules $\pi : \hat{\Omega}_{\mathcal{O}/k}(\log \boldsymbol{x}) \hookrightarrow \hat{\Omega}_{\mathcal{O}'/k}(\log \boldsymbol{x}')$ satisfies

$$\pi(N_{\mathcal{A}}^{j}) \subset N_{\mathcal{A}'}^{\ell} \quad \text{for} \quad 0 \leq j \leq \ell \ ,$$

and

$$\pi(N_{\mathcal{A}}^{k}) \subset N_{\mathcal{A}'}^{k} \quad \text{for} \quad \ell+1 \leq k \leq n-r \ .$$

In fact, we have that

$$\pi_{|N_{\mathcal{A}}^{\ell}} : N_{\mathcal{A}}^{\ell} \to N_{\mathcal{A}'}^{\ell}$$

is a $R_{\mathcal{A}}^{\ell}$-module monomorphism.

Chapter 3

Maximal rational rank: the combinatorial case

In this chapter we treat the case of valuations of maximal rational rank $r = \operatorname{rat.rk}(\nu) = \operatorname{tr.deg}(K/k)$. Let $\mathcal{A} = (\mathcal{O}, \boldsymbol{x})$ be a parameterized regular local model for K, ν. Recall that in this case we have

$$\hat{\mathcal{O}} \simeq R^0_{\mathcal{A}} \quad \text{and} \quad \Omega_{\mathcal{O}/k}(\log \boldsymbol{x}) \otimes_{\mathcal{O}} \hat{\mathcal{O}} \simeq N^0_{\mathcal{A}} \ .$$

The transformations of parameterized regular local models allowed in this case are the 0-nested transformations. Such a transformation is a finite composition

$$\mathcal{A} = \mathcal{A}_0 \xrightarrow{\tau_0} \mathcal{A}_1 \xrightarrow{\tau_1} \cdots \xrightarrow{\tau_{t-1}} \mathcal{A}_t = \mathcal{A}'$$

where each

$$\mathcal{A}_k \xrightarrow{\tau_k} \mathcal{A}_{k+1}$$

is a combinatorial blow-up $\tau_k = \theta_{i_k j_k}$ with $1 \leq i_k < j_k \leq r$. The inclusion of local rings is given by

$$\begin{array}{rcl} R^0_{\mathcal{A}} & \longrightarrow & R^0_{\mathcal{A}'} \\ x_i & \longmapsto & \boldsymbol{x'}^{\boldsymbol{c}_i} = {x'}_1^{c_{i1}} \cdots {x'}_r^{c_{ir}} \ , \end{array} \tag{3.1}$$

where the matrix $C := (c_{ij})$ is an invertible matrix of non-negative integers with determinant 1. Note that this homomorphism preserves the value of the monomials, it means

$$\nu(\boldsymbol{x}^{\boldsymbol{a}}) = \nu(\boldsymbol{x'}^{\boldsymbol{a}'}) \ ,$$

where $\boldsymbol{a}' = \boldsymbol{a}C$. The divisibility relation is also maintained:

$$\boldsymbol{x}^{\boldsymbol{a}} | \boldsymbol{x}^{\boldsymbol{b}} \Rightarrow \boldsymbol{x'}^{\boldsymbol{a}'} | \boldsymbol{x'}^{\boldsymbol{b}'} \ .$$

Taking differentials in (3.1) we obtain

$$\frac{dx_i}{x_i} = c_{i1} \frac{dx'_1}{x'_1} + \cdots + c_{ir} \frac{dx'_r}{x'_r}$$

for $1 \leq i \leq r$. Therefore we have that the $R^0_{\mathcal{A}}$-homomorphism between the modules of differentials is given by

$$\begin{array}{ccc} N^0_{\mathcal{A}} & \longrightarrow & N^0_{\mathcal{A}'} \\ \sum a_i \dfrac{dx_i}{x_i} & \longmapsto & \sum a'_i \dfrac{dx'_i}{x'_i} \end{array} ,$$

where

$$a'_i = c_{i1}a_1 + \cdots + c_{ir}a_r$$

for $1 \leq i \leq r$. It is in fact a monomorphism since the matrix C is invertible.

Before prove Theorem 2 in this case, we present two useful lemmas which we will use frequently. Let $\mathcal{L} = \{F_i \mid i \in I\} \subset R^0_{\mathcal{A}}$ be a list of elements of $R^0_{\mathcal{A}}$. We say the list $\mathcal{L}$ is *simple in* $\mathcal{A}$ if for all pair of elements $F_i, F_j \in \mathcal{L}$ we have that $\nu(F_i) \leq \nu(F_j)$ if and only if F_i divides F_j.

Lemma 1. *Let $\mathcal{L} \subset R^0_{\mathcal{A}}$ be a finite list of monomials. There is a 0-nested transformation $\mathcal{A} \to \mathcal{A}'$ such that $\mathcal{L}$ is simple in $\mathcal{A}'$.*

Proof. Since the divisibility relation remains stable after performing a combinatorial blow-up in the independent variables, it is enough to prove the statement for lists with just two monomials. Consider $\mathcal{L} = \{\boldsymbol{x}^{\boldsymbol{a}}, \boldsymbol{x}^{\boldsymbol{b}}\}$ and put $\boldsymbol{c} = \boldsymbol{a} - \boldsymbol{b}$. We use the following invariants

$$M := \max\big\{0, c_1, \ldots, c_r\big\} \ , \quad m := \min\big\{0, c_1, \ldots, c_r\big\},$$

$$T := \#\big\{i \ : \ c_i = M\big\} \ , \quad t := \#\big\{i \ : \ c_i = m\big\} \ , \quad \delta := \big(-Mm, T + t\big).$$

If the first coordinate of δ is 0 the list $\mathcal{L}$ is simple. If it is not the case perform a blow-up θ_{ij} such that $c_i = m$ and $c_j = M$. Without loss of generality we can suppose $\nu(x_i) < \nu(x_j)$. At the center of ν in the new variety obtained we have a system of coordinates $\boldsymbol{x}'$ which only differs from $\boldsymbol{x}$ in the j-th variable, which is $x'_j = x_j/x_i$. The exponents $\boldsymbol{a}$ and $\boldsymbol{b}$ becomes $\boldsymbol{a}'$ and $\boldsymbol{b}'$ respectively, where only the i-th coordinate is modified. They are $a'_i = a_i + a_j$ and $b'_i = b_i + b_j$. The same thing happens with $\boldsymbol{c}'$. We have $m = c_i < 0 < c_j = M$, thus $m < c'_i = c_i + c_j < M$. Then $\delta' <_{lex} \delta$, so iterating we get the desired result. □

For lists with infinitely many monomials we have the following statement:

Lemma 2. *Let $\mathcal{L} \subset R^0_{\mathcal{A}}$ be an infinite list of monomials. There is a 0-nested transformation $\mathcal{A} \to \mathcal{A}'$ such that in $\mathcal{A}'$ every monomial of $\mathcal{L}$ is divided by the monomial with lowest value.*

Proof. Consider the ideal $I \subset \mathcal{O}$ generated by the elements of $\mathcal{L}$. Since $\mathcal{O}$ is Noetherian, I is finitely generated. It is enough to apply Lemma 3 to $\mathcal{L}'$, where $\mathcal{L}'$ is the list formed by a finite system of generators of I. □

The Local Uniformization of formal functions in the maximal rational case is a corollary of Lemma 2. Let us see in detail. Take a formal function

$$F = \sum_{\boldsymbol{a} \in \mathbb{Z}^r_{\geq 0}} F_{\boldsymbol{a}} \boldsymbol{x}^{\boldsymbol{a}} \in R^0_{\mathcal{A}}$$

and let $\mathcal{L}_F$ be the list formed by its monomials

$$\mathcal{L}_F := \left\{ \boldsymbol{x}^{\boldsymbol{a}} \mid F_{\boldsymbol{a}} \neq 0 \right\} .$$

Applying Lemma 4 to $\mathcal{L}_F$ we obtain a new parametrized regular local model $\mathcal{B}$ in which F has the form

$$F = \boldsymbol{x}'^{\boldsymbol{t}} U \in R^0_{\mathcal{B}} ,$$

where $U \in R^0_{\mathcal{B}}$ is a unit. We say that we have *monomialized* the formal function F.

Now we can improve the statements of Lemmas 1 and 2.

Lemma 3. *Let $\mathcal{L} \subset R^0_{\mathcal{A}}$ be a finite list. There is a 0-nested transformation $\mathcal{A} \to \mathcal{A}'$ such that $\mathcal{L}$ is simple in $\mathcal{A}'$.*

Proof. First, we monomialize each element of the list using Lemma 2, so each element of the list becomes a monomial times a unit. Then we apply Lemma 1 to the list of such monomials. □

Lemma 4. *Let $\mathcal{L} \subset R^0_{\mathcal{A}}$ be an infinite list. There is a 0-nested transformation $\mathcal{A} \to \mathcal{A}'$ such that in $\mathcal{A}'$ every monomial of $\mathcal{L}$ is divided by the monomial with lowest value.*

Proof. We just need to apply Lemma 2 to the list formed by all the monomials appearing in the elements of $\mathcal{L}$. □

Using these lemmas we are able to prove Theorem 2 in the maximal rational case.

Let $\mathcal{F}$ be a rational codimension one foliation of K/k and take a 1-form

$$\omega = \sum a_i \frac{dx_i}{x_i} \in F_{\mathcal{A}} \subset N^0_{\mathcal{A}} .$$

Consider the list

$$\mathcal{L}_{\omega,\mathcal{A}} := \left\{ a_1, a_2, \ldots, a_r \right\} \subset R^0_{\mathcal{A}} .$$

Using Lemma 3 we can determine a 0-nested transformation $\mathcal{A} \to \mathcal{B}$ such that $\mathcal{L}_{\omega,\mathcal{A}}$ is simple in $\mathcal{B}$. As we have just explained, in $\mathcal{B}$ we have

$$\omega = \sum a'_i \frac{dx'_i}{x'_i} \in \mathcal{F}_{\mathcal{B}} \subset N^0_{\mathcal{B}}$$

where the coefficients $a'_i \in R^0_{\mathcal{B}}$ are an invertible linear combination of the coefficients $a_i \in R^0_{\mathcal{A}}$. Therefore the list

$$\mathcal{L}_{\omega,\mathcal{B}} = \left\{ a'_1, a'_2, \ldots, a'_r \right\}$$

is also simple, so we can factorize the coefficient with lower value and obtain an expression

$$\omega = \boldsymbol{x}'^{\boldsymbol{t}} \sum_{i=1}^{r} \tilde{a}_i \frac{dx'_i}{x'_i} \in \mathcal{F}_{\mathcal{B}} \subset N^0_{\mathcal{B}}$$

where at least one of the coefficients $\tilde{a}_i$ is a unit. We have that the 1-form

$$\tilde{\omega} := \frac{1}{\boldsymbol{x}'^{\boldsymbol{t}}} \omega = \sum_{i=1}^{r} \tilde{a}_i \frac{dx'_i}{x'_i}$$

belongs to $\mathcal{F}_{\mathcal{B}}$ and it is $\boldsymbol{x}'$-log-elementary, hence $\mathcal{F}$ is $\mathcal{B}$-final.

Remark 10. Note that we have not use neither the integrability condition nor the algebraic nature of the coefficients. It means that in the maximal rational case we have proved a more general result:

> "Given a 1-form $\omega \in N^0_{\mathcal{A}}$ there is a 0-nested transformation $\mathcal{A} \to \mathcal{B}$ such that in $\mathcal{B}$ we have $\omega = \boldsymbol{x}'^{\boldsymbol{t}}\tilde{\omega}$ being $\tilde{\omega}$ log-elementary."

Remark 11. Note that the results in this section can be used in the case $r = \operatorname{rat.rk}(\nu) < \operatorname{tr.deg}(K/k) = n$ if we restrict ourselves to elements of $R^0_{\mathcal{B}} \subsetneq R^{n-r}_{\mathcal{B}}$ and $N^0_{\mathcal{B}} \subsetneq N^{n-r}_{\mathcal{B}}$.

3.1 Existence of Puiseux's packages

Using the results of this chapter we are now able to prove the existence of Puiseux's packages.

Proposition 5. *Let $\mathcal{A}$ be a parameterized regular local model of K, ν. Given a dependent variable y_ℓ there are ℓ-Puiseux's packages.*

Proof. Consider the l-th contact rational function

$$\phi_\ell = \frac{\boldsymbol{x}^{\boldsymbol{q}} y_\ell^d}{\boldsymbol{x}^{\boldsymbol{t}}}.$$

Applying Lemma 3 to the list $\{\boldsymbol{x}^{\boldsymbol{q}}, \boldsymbol{x}^{\boldsymbol{t}}\}$, we can reduce to the case $\boldsymbol{q} = 0$. We use as invariant $\delta = (d, \sum t_i)$.

Suppose $\delta = (1,1)$ and let i_0 be the only index such that $t_{i_0} \neq 0$. We have that $\nu(y_\ell) = \nu(x_{i_0})$ so the blow-up $\theta_{i_0,s}$ is a blow up with translation hence

$$\theta_{i_0,s} : \mathcal{A}_0 \longrightarrow \mathcal{A}_1$$

is a ℓ-Puiseux's package.

Suppose that $\delta = (1, M)$ with $M > 1$ and let i_0 be an index such that $t_{i_0} \neq 0$. In this case $\theta_{i_0,s}$ is combinatorial. In the new coordinates we have $\delta = (1, M-1)$ so iterating we reach the previous situation.

Finally suppose that $d > 1$. If there is an index i_0 such that $t_{i_0} \neq 0$ and $\nu(y_\ell) < \nu(x_{i_0})$ then after the combinatorial blow-up $\theta_{i_0,s}$ the invariant becomes $\delta = (d - p_{i_0}, M)$. On the other hand, if $\nu(y_\ell) > \nu(x_i)$ for all index i with $t_i \neq 0$ let i_0 be the first index such that $d \leq \sum_{i=1}^{i_0} t_i$. Perform the blow-up with center (x_1, x_{i_0}), then the one with center (x_2, x_{i_0}) and continue until (x_{i_0-1}, x_{i_0}) (exclude the blow-ups corresponding with independent variables with $t_i = 0$). Perform the blow-up with center (x_{i_0}, y_ℓ). After this sequence of combinatorial blow-ups the invariant is $\delta = (d, M')$ with $M' < M$ since the new exponent of the variable x_{i_0} is $0 < t'_{i_0} = \sum_{i=1}^{i_0} t_i - d < t_{i_0}$. We observe that in both cases the invariant δ decreases for the lexicographic order, so iterating we reach the case $d = 1$. $\square$

Chapter 4

Explicit value and truncated statements. Induction structure

In this chapter introduce the notions of γ-final formal functions and 1-forms and state local uniformization theorems in a value-truncated version. The local uniformization of foliations will be a consequence of this truncated version.

Let $\mathcal{A} = \big(\mathcal{O}; (\boldsymbol{x}, \boldsymbol{y})\big)$ be a parameterized regular local model for K, ν and fix an integer ℓ, $0 \leq \ell \leq n - r$.

4.1 Explicit value and γ-final 1-forms

Given a formal function $F \in R^{\ell}_{\mathcal{A}}$ we can write it as a power series in the independent variables

$$F = \sum_{I \in \mathbb{Z}^r_{\geq 0}} F_I(\boldsymbol{y}) \boldsymbol{x}^I \ , \quad F_I(\boldsymbol{y}) \in k[[\boldsymbol{y}]] \ .$$

We define *the explicit value of* F by

$$\nu_{\mathcal{A}}(F) := \min \left\{ \nu(\boldsymbol{x}^I) \mid F_I(\boldsymbol{y}) \neq 0 \right\} \ ,$$

where we establish $\nu_{\mathcal{A}}(0) = \infty$.

Definition 15. Given $\gamma \in \Gamma$ a formal function $F \in R^{\ell}_{\mathcal{A}}$ is γ-*final* if one of the following properties is satisfied:

1. $\nu_{\mathcal{A}}(F) \leq \gamma$ and F can be written as
$$F = \boldsymbol{x}^{\boldsymbol{t}} \tilde{F} + \bar{F}$$
where $\tilde{F}$ is a unit and $\nu_{\mathcal{A}}(\bar{F}) > \nu_{\mathcal{A}}(F) = \nu(\boldsymbol{x}^{\boldsymbol{t}})$. In this case we say F *is* γ-*final dominant*;

2. $\nu_{\mathcal{A}}(F) > \gamma$. In this case we say F *is* γ-*final recessive.*

Remark 12. Let F be a γ-final dominant formal function. For every valuation $\hat{\nu}$ defined on $\hat{\mathcal{O}}$ extending ν, we have that $\hat{\nu}(F) = \nu_{\mathcal{A}}(F) = \nu(\boldsymbol{x}^{\boldsymbol{t}})$.

While the value $\nu(\phi)$ of a rational function $\phi \in K$ is stable under birational transformations, the explicit value can increase by means of a ℓ-nested transformation:

Lemma 5. *Let $F \in R^{\ell}_{\mathcal{A}}$ be a formal function and let $\pi : \mathcal{A} \longrightarrow \mathcal{B}$ be a ℓ-nested transformation. We have that*

$$\nu_{\mathcal{A}}(F) \leq \nu_{\mathcal{B}}(F) \ .$$

In particular we have

- *if F is γ-final dominant in $\mathcal{A}$, it is also γ-final dominant in $\mathcal{B}$ and $\nu_{\mathcal{A}}(F) = \nu_{\mathcal{B}}(F)$.*
- *if F is γ-final recessive in $\mathcal{A}$, it is also γ-final recessive in $\mathcal{B}$.*

Now we extend the definition of explicit value to differential forms. Given a 1-form with formal coefficients

$$\omega = \sum_{i=1}^{r} a_i \frac{dx_i}{x_i} + \sum_{j=r+1}^{n} a_j dy_j \in N^{\ell}_{\mathcal{A}}$$

we define *the explicit value of* ω by

$$\nu_{\mathcal{A}}(\omega) := \min \{\nu_{\mathcal{A}}(a_1), \ldots, \nu_{\mathcal{A}}(a_r), \nu_{\mathcal{A}}(b_1), \ldots, \nu_{\mathcal{A}}(b_\ell)\} \ ,$$

where we establish $\nu_{\mathcal{A}}(0) = \infty$. As in the case of functions, writing

$$\omega = \sum_{I \in \mathbb{Z}^r_{\geq 0}} \boldsymbol{x}^I \omega_I(\boldsymbol{y}) \ ,$$

where each $\omega_I(\boldsymbol{y})$ is a 1-form whose coefficients are series in the dependent variables, we have that

$$\nu_{\mathcal{A}}(\omega) = \min \left\{\nu(\boldsymbol{x}^I) \mid \omega_I(\boldsymbol{y}) \neq 0\right\} \ .$$

In the same way we define the explicit value for elements of $\wedge^2 N^{\ell}_{\mathcal{A}}$ and $\wedge^3 N^{\ell}_{\mathcal{A}}$, where $\wedge^p$ denotes the p-th exterior power.

In the module $N^{\ell}_{\mathcal{A}}$ the log-elementary forms play the role of the units in $R^{\ell}_{\mathcal{A}}$. Recall that a 1-form

$$\omega = \sum_{i=1}^{r} a_i \frac{dx_i}{x_i} + \sum_{j=1}^{\ell} b_j dy_j \in N^{\ell}_{\mathcal{A}}$$

is log-elementary if at least one of the coefficients $a_1, a_2, \ldots, a_r \in R^{\ell}_{\mathcal{A}}$ is a unit.

Definition 16. Given $\gamma \in \Gamma$ a 1-form $\omega \in N^{\ell}_{\mathcal{A}}$ is *γ-final* if one of the following properties is satisfied

1. $\nu_{\mathcal{A}}(\omega) \leq \gamma$ and ω can be written as

$$\omega = \boldsymbol{x}^{\boldsymbol{t}} \tilde{\omega} + \bar{\omega}$$

where $\tilde{\omega}$ is log-elementary and $\nu_{\mathcal{A}}(\bar{\omega}) > \nu_{\mathcal{A}}(\omega) = \nu(\boldsymbol{x}^{\boldsymbol{t}})$. In this case we say *$\omega$ is γ-final dominant*;

2. $\nu_{\mathcal{A}}(\omega) > \gamma$. In this case we say ω *is γ-final recessive.*

Let $\omega \in N^{\ell}_{\mathcal{A}}$ be a 1-form and write it in the form

$$\omega = \boldsymbol{x}^{\boldsymbol{t}}\tilde{\omega} + \bar{\omega}$$

where

$$\nu_{\mathcal{A}}(\omega) = \nu(\boldsymbol{x}^{\boldsymbol{t}}) \ , \quad \nu_{\mathcal{A}}(\tilde{\omega}) = 0 \quad \text{and} \quad \nu_{\mathcal{A}}(\bar{\omega}) > \nu(\boldsymbol{x}^{\boldsymbol{t}}) \ .$$

For each index $1 \leq i \leq r$ denote $\alpha_i = a_i(\mathbf{0}) \in k$, where $a_i \in R^{\ell}_{\mathcal{A}}$ are the coefficients of $\tilde{\omega}$

$$\tilde{\omega} = \sum_{i=1}^{r} a_i \frac{dx_i}{x_i} + \sum_{j=1}^{\ell} b_j dy_j \ .$$

We define the $\mathcal{A}$*-initial part of* ω by

$$\mathrm{in}_{\mathcal{A}}(\omega) = \boldsymbol{x}^{\boldsymbol{t}} \sum_{i=1}^{r} \alpha_i \frac{dx_i}{x_i} \ .$$

Remark 13. Consider a 1-form $\omega \in N^{\ell}_{\mathcal{A}}$. We have that ω is $\nu_{\mathcal{A}}(\omega)$-final dominant if and only if $\mathrm{in}_{\mathcal{A}}(\omega) \neq 0$.

As in the case of functions, the explicit value of a 1-form can increase by means of a ℓ-nested transformation:

Lemma 6. *Let $\omega \in N^{\ell}_{\mathcal{A}}$ be a formal 1-form and let $\pi : \mathcal{A} \longrightarrow \mathcal{B}$ be a ℓ-nested transformation. We have that*

$$\nu_{\mathcal{A}}(\omega) \leq \nu_{\mathcal{B}}(\omega) \ .$$

In particular we have

1. *if ω is γ-final dominant in $\mathcal{A}$, it is also γ-final dominant in $\mathcal{B}$ and $\nu_{\mathcal{A}}(\omega) = \nu_{\mathcal{B}}(\omega)$.*

2. *if ω is γ-final recessive in $\mathcal{A}$, it is also γ-final recessive in $\mathcal{B}$.*

This lemma is a consequence of the definitions. We do not detail the proof, but the key is the following remark:

Remark 14. Let $\omega \in N^{\ell}_{\mathcal{A}}$ be a $\nu_{\mathcal{A}}(\omega)$-final dominant 1-form with $\mathcal{A}$-initial part given by

$$\mathrm{in}_{\mathcal{A}}(\omega) = \boldsymbol{x}^{\boldsymbol{t}} \sum_{i=1}^{r} \alpha_i \frac{dx_i}{x_i} \ .$$

Consider a ℓ-nested transformation $\pi : \mathcal{A} \longrightarrow \mathcal{B}$. We have that

$$\mathrm{in}_{\mathcal{B}}(\omega) = \boldsymbol{x}'^{\boldsymbol{t}'} \sum_{i=1}^{r} \alpha'_i \frac{dx'_i}{x'_i} \ ,$$

where $\boldsymbol{x}'$ are the independent variables in $\mathcal{B}$, $\nu(\boldsymbol{x}'^{\boldsymbol{t}'}) = \nu(\boldsymbol{x}^{\boldsymbol{t}})$ and

$$(\alpha'_1, \alpha'_2, \ldots, \alpha'_r) = (\alpha_1, \alpha_2, \ldots, \alpha_r) C_\pi$$

where C_π is a non-zero matrix of non-negative integers.

Let

$$d : \mathcal{O} \longrightarrow \Omega_{\mathcal{O}/k}(\log \boldsymbol{x})$$

be the map obtained by composition of the exterior derivative $\mathcal{O} \to \Omega_{\mathcal{O}/k}$ with the inclusion $\Omega_{\mathcal{O}/k} \to \Omega_{\mathcal{O}/k}(\log \boldsymbol{x})$ given in (1.1). Tensoring by $\hat{\mathcal{O}}$ we obtain

$$d \otimes_{\mathcal{O}} 1 : \mathcal{O} \otimes_{\mathcal{O}} \hat{\mathcal{O}} \longrightarrow \Omega_{\mathcal{O}/k}(\log \boldsymbol{x}) \otimes_{\mathcal{O}} \hat{\mathcal{O}} \ .$$

By abuse of notation we denote the map $d \otimes_{\mathcal{O}} 1$ just by d. Explicitly, we have just defined the map

$$\begin{array}{rcl} d : R_{\mathcal{A}}^{n-r} & \longrightarrow & N_{\mathcal{A}}^{n-r} \\ F & \longmapsto & \displaystyle\sum_{i=1}^{r} x_i \frac{\partial F}{\partial x_i} \frac{dx_i}{x_i} + \sum_{j=1}^{n-r} \frac{\partial F}{\partial y_j} dy_j \end{array}$$

Note that for any index ℓ, $0 \le \ell \le n-r$, we have

$$d(R_{\mathcal{A}}^{\ell}) \subset N_{\mathcal{A}}^{\ell} \ .$$

Proposition 6. *Let $F \in R_{\mathcal{A}}^{\ell}$ be formal function which is not a unit. We have that*

$$\nu_{\mathcal{A}}(F) = \nu_{\mathcal{A}}(dF) \ .$$

In addition, fixed a value $\gamma \in \Gamma$, we have that F is γ-final dominant (recessive) if and only if $dF \in N_{\mathcal{A}}^{\ell}$ is γ-final dominant (recessive).

Proof. Write F as a series in monomials in the independent variables:

$$F = \sum_{T \in \mathbb{Z}_{\ge 0}^r} F_{\boldsymbol{T}}(\boldsymbol{y}) \boldsymbol{x}^{\boldsymbol{T}}.$$

We have

$$dF = \sum_{T \in \mathbb{Z}_{\ge 0}^r} \boldsymbol{x}^{\boldsymbol{T}} \left(F_{\boldsymbol{T}}(\boldsymbol{y}) \sum_{i=1}^{r} T_i \frac{dx_i}{x_i} + \sum_{i=r+1}^{n} \frac{\partial F_{\boldsymbol{T}}}{\partial y_i}(\boldsymbol{y}) dy_i \right) \ .$$

The result follows from the equivalences

$$F_{\boldsymbol{T}}(\boldsymbol{y}) = 0 \Longleftrightarrow F_{\boldsymbol{T}}(\boldsymbol{y}) \sum_{i=1}^{r} T_i \frac{dx_i}{x_i} + \sum_{i=r+1}^{n} \frac{\partial F_{\boldsymbol{T}}}{\partial y_i}(\boldsymbol{y}) dy_i = 0$$

and

$$F_{\boldsymbol{T}}(\boldsymbol{y}) \text{ is a unit } \Longleftrightarrow F_{\boldsymbol{T}}(\boldsymbol{y}) \sum_{i=1}^{r} T_i \frac{dx_i}{x_i} + \sum_{i=r+1}^{n} \frac{\partial F_{\boldsymbol{T}}}{\partial y_i}(\boldsymbol{y}) dy_i \text{ is log-elementary.}$$

□

Proceeding as before we define the map

$$\Omega_{\mathcal{O}/k}(\log \boldsymbol{x}) \otimes_{\mathcal{O}} \hat{\mathcal{O}} \to \wedge^2 \, \Omega_{\mathcal{O}/k}(\log \boldsymbol{x}) \otimes_{\mathcal{O}} \hat{\mathcal{O}} \ ,$$

and we denote it also by d. Again, note that for any index ℓ, $0 \leq \ell \leq n-r$ we have

$$d(N_{\mathcal{A}}^{\ell}) \subset \wedge^2 N_{\mathcal{A}}^{\ell} \ .$$

A direct computation shows that for $\omega \in N_{\mathcal{A}}^{\ell}$ we have

$$\nu_{\mathcal{A}}(\omega) \leq \nu_{\mathcal{A}}(d\omega) \ .$$

We state now the corresponding notions for pairs $(\omega, F) \in N_{\mathcal{A}}^{\ell} \times R_{\mathcal{A}}^{\ell}$. Note that this Cartesian product has naturally structure of $R_{\mathcal{A}}^{\ell}$-module. Given a pair $(\omega, F) \in N_{\mathcal{A}}^{\ell} \times R_{\mathcal{A}}^{\ell}$ we say that

$$\nu_{\mathcal{A}}(\omega, F) := \min \{\nu_{\mathcal{A}}(\omega), \nu_{\mathcal{A}}(F)\}$$

is *the explicit value of* (ω, F). We establish $\nu_{\mathcal{A}}(0,0) := \infty$.

Definition 17. Given $\gamma \in \Gamma$ a pair $(\omega, F) \in N_{\mathcal{A}}^{\ell} \times R_{\mathcal{A}}^{\ell}$ is *γ-final* if one of the following properties is satisfied:

1. $\nu_{\mathcal{A}}(\omega, F) \leq \gamma$ and ω and F are both $\nu_{\mathcal{A}}(\omega, F)$-final. In this case we say (ω, F) *is γ-final dominant*;
2. $\nu_{\mathcal{A}}(\omega, F) > \gamma$. In this case we say (ω, F) *is γ-final recessive.*

Remark 15. Note that the definition of a γ-final dominant pair is slightly different that the corresponding to functions or 1-forms. The following is an equivalent definition:

> A pair $(\omega, F) \in N_{\mathcal{A}}^{\ell} \times R_{\mathcal{A}}^{\ell}$ is γ-final dominant if $\nu_{\mathcal{A}}(\omega, F) \leq \gamma$ and it can be written as
>
> $$(\omega, F) = \boldsymbol{x}^{\boldsymbol{t}}(\tilde{\omega}, \tilde{F}) + (\bar{\omega}, \bar{f})$$
>
> where $\nu_{\mathcal{A}}(\bar{\omega}, \bar{f}) > \nu_{\mathcal{A}}(\omega, F) = \nu(\boldsymbol{x}^{\boldsymbol{t}})$ and one of the following options is satisfied:
>
> 1. $\tilde{\omega}$ is log-elementary and $\tilde{F}$ is a unit;
> 2. $\tilde{\omega}$ is log-elementary and $\tilde{F} = 0$;
> 3. $\tilde{\omega} = 0$ and $\tilde{F}$ is a unit.

Remark 16. Note that if $\omega \in N_{\mathcal{A}}^{\ell}$ and $F \in R_{\mathcal{A}}^{\ell}$ are both γ-final then the pair $(\omega, F) \in N_{\mathcal{A}}^{\ell} \times R_{\mathcal{A}}^{\ell}$ is also γ-final. However, the opposite is not necessarily true: it happens when $\nu_{\mathcal{A}}(\omega, F) = \nu_{\mathcal{A}}(\omega) = \nu_{\mathcal{A}}(F) < \gamma$ but only one of the terms of the pair is $\nu_{\mathcal{A}}(\omega, F)$-final dominant.

Given a ℓ-nested transformation $\mathcal{A} \to \mathcal{A}'$ it is well defined the $R_{\mathcal{A}}^{\ell}$-module monomorphism

$$N_{\mathcal{A}}^{\ell} \times R_{\mathcal{A}}^{\ell} \longrightarrow N_{\mathcal{A}'}^{\ell} \times R_{\mathcal{A}'}^{\ell} \ .$$

As in previous cases, the explicit value of a pair can increase by means of a ℓ-nested transformation:

Lemma 7. *Let $(\omega, F) \in N_{\mathcal{A}}^{\ell} \times R_{\mathcal{A}}^{\ell}$ be a pair and let $\pi : \mathcal{A} \longrightarrow \mathcal{B}$ be a ℓ-nested transformation. We have that*

$$\nu_{\mathcal{A}}(\omega, F) \leq \nu_{\mathcal{B}}(\omega, F) \ .$$

In particular we have

1. *if (ω, F) is γ-final dominant in $\mathcal{A}$, it is also γ-final dominant in $\mathcal{B}$ and $\nu_{\mathcal{A}}(\omega, F) = \nu_{\mathcal{B}}(\omega, F)$.*

2. *if (ω, F) is γ-final recessive in $\mathcal{A}$, it is also γ-final recessive in $\mathcal{B}$.*

The following lemma provides a simple but powerful tool which allows to "push right" non dominant objects:

Lemma 8. *There is a ℓ-nested transformation $\Psi_\ell : \mathcal{A} \to \mathcal{B}$ such that for any object (function, 1-form or pair) ψ we have:*

$$\psi \text{ is not dominant in } \mathcal{A} \Longrightarrow \nu_{\mathcal{A}}(\psi) < \nu_{\mathcal{B}}(\psi) \ .$$

Proof. We have to perform Puiseux's packages with respect to all dependent variables $y_1, y_2, \ldots, y_\ell$. So we can take

$$\Psi_\ell := \pi_\ell \circ \cdots \circ \pi_1$$

where π_j is a j-Puiseux's package. □

4.2 Truncated Local Uniformization statements

The following statement is a refinement of Theorem III. It is the key in the proof of Theorem 2.

Theorem 3 (Truncated Local Uniformization of formal differential 1-forms)**.** *Let $\mathcal{A}$ be a parameterized regular local model for K, ν and let ℓ be an index $0 \le \ell \le n - r$. Given a 1-form $\omega \in N^\ell_{\mathcal{A}}$ and a value $\gamma \in \Gamma$, if $\nu_{\mathcal{A}}(\omega \wedge d\omega) \ge 2\gamma$ then there exists a ℓ-nested transformation $\mathcal{A} \to \mathcal{B}$ such that ω is γ-final in $\mathcal{B}$.*

Taking into account that for a formal function F we have that $d(dF) \equiv 0$, Theorem 4 and Proposition 6 implies the following statement.

Theorem 4 (Truncated Local Uniformization of formal functions)**.** *Let $\mathcal{A}$ be a parameterized regular local model for K, ν and let ℓ be an index $0 \le \ell \le n - r$. Given a formal function $F \in R^\ell_{\mathcal{A}}$ and a value $\gamma \in \Gamma$, there exists a ℓ-nested transformation $\mathcal{A} \to \mathcal{B}$ such that F is γ-final in $\mathcal{B}$.*

We have also the corresponding statement for pairs:

Theorem 5. *Let $\mathcal{A}$ be a parameterized regular local model for K, ν and let ℓ be an index $0 \le \ell \le n - r$. Given a pair $(\omega, F) \in N^\ell_{\mathcal{A}} \times R^\ell_{\mathcal{A}}$ and a value $\gamma \in \Gamma$, if $\nu_{\mathcal{A}}(\omega \wedge d\omega) \ge 2\gamma$ then there exists a ℓ-nested transformation $\mathcal{A} \to \mathcal{B}$ such that (ω, F) is γ-final in $\mathcal{B}$.*

We have that Theorem 5 is also a consequence of Theorem 3. Thanks to Theorems 3 and 4 and Lemmas 6 and 5 we can obtain a pair whose both terms (1-form and function) are γ-final, hence the pair is also γ-final.

4.3 Induction procedure

In the statements of Theorems 3, 4 and 5 appears a parameter ℓ, $0 \leq \ell \leq n-r$. Let us refer to these theorems by $T_3(\ell)$, $T_4(\ell)$ and $T_5(\ell)$ respectively to indicate a fixed parameter ℓ. As we said in the previous section we have that

$$T_3(\ell) \Rightarrow T_4(\ell) \quad \text{and} \quad T_3(\ell) \Rightarrow T_5(\ell)\ .$$

Note also that for $i = 3, 4, 5$ we have

$$T_i(\ell) \Rightarrow T_i(\ell') \quad \text{for all} \quad 0 \leq \ell' < \ell \leq n-r\ ,$$

and in particular

$$T_i(n-r) \Leftrightarrow T_i\ .$$

Our goal is to prove Theorem 3. In Chapter 3 we proved $T_3(0)$. In Chapters 6 and 7 we conclude the proof of Theorem 3 by proving the induction step

$$T_3(\ell) \Rightarrow T_3(\ell+1)\ ,$$

so in particular we will also prove Theorems 4 and 5. However, in Chapter 5 we detail the proof of

$$T_4(\ell) \Rightarrow T_4(\ell+1)\ ,$$

since we will use that proof as a guide for the next chapters.

Chapter 5

Truncated Local Uniformization of functions

Let $\mathcal{A} = \big(\mathcal{O}, (\boldsymbol{x}, \boldsymbol{y})\big)$ be a parameterized regular local model of K, ν. Fix an index ℓ, $0 \leq \ell \leq n - r - 1$ and a value $\gamma \in \Gamma$.

In this chapter we consider a function $F \in R_{\mathcal{A}}^{\ell+1}$ which is not γ-final. We assume $T_4(\ell)$ and we will show $T_4(\ell + 1)$, that is, there is a $(\ell + 1)$-nested transformation $\mathcal{A} \to \mathcal{B}$ such that F is γ-final in $\mathcal{B}$.

5.1 Truncated preparation of a function

Let us denote the dependent variable $y_{\ell+1}$ by z. Write F as a power series in the dependent variable z:

$$F = \sum_{k=0}^{\infty} z^k F_k \ , \quad F_k \in R_{\mathcal{A}}^{\ell} \text{ for } k \geq 0 \ .$$

For each $k \geq 0$ denote by $\phi_k(F; \mathcal{A}) \in \Gamma$ the explicit value

$$\phi_k(F; \mathcal{A}) := \phi_k(F; \mathcal{A}) \ .$$

The *Cloud of Points of F* is the discrete subset

$$\mathrm{CL}(F; \mathcal{A}) := \{(\phi_k, k) \mid k = 0, 1, \dots\} \ .$$

Note that

$$F \neq 0 \Rightarrow \mathrm{CL}(F; \mathcal{A}) \neq \emptyset \ .$$

We also use the *Dominant Cloud of Points of F*

$$\mathrm{DomCL}(F; \mathcal{A}) := \{(\beta_k, k) \in \mathrm{CL}(F; \mathcal{A}) \mid F_k \text{ is dominant}\} \ .$$

Note that $\mathrm{DomCL}(F; \mathcal{A})$ can be empty. In Figure 5.1 we can see an example in which the points (ϕ_k, k) are represented with black and white circles, corresponding to dominant and non-dominant levels respectively. Given a value $\sigma \in \Gamma$, we define the truncated polygons

$$\mathcal{N}(F; \mathcal{A}, \sigma) \quad \text{and} \quad \mathrm{Dom}\mathcal{N}(F; \mathcal{A}, \sigma)$$

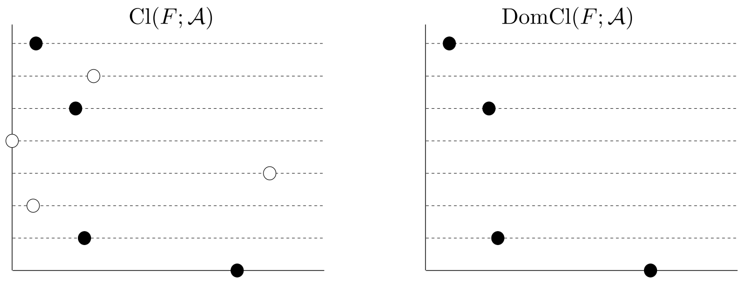

Figure 5.1: The Cloud of Points and the Dominant Cloud of Points

to be the respective positively convex hulls of

$$\{(0, \sigma/\nu(z)), (\sigma, 0)\} \cup \mathrm{CL}(F; \mathcal{A})$$

and

$$\{(0, \sigma/\nu(z)), (\sigma, 0)\} \cup \mathrm{DomCL}(F; \mathcal{A}) \ .$$

Note that for any $\sigma \in \Gamma$ we have that

$$\mathcal{N}(F; \mathcal{A}, \sigma) \supset \mathrm{Dom}\,\mathcal{N}(F; \mathcal{A}, \sigma) \ .$$

In Figure 5.2 we can see the truncated polygons corresponding to the clouds of points represented in Figure 5.1. Note that in this example $\mathrm{Dom}\,\mathcal{N}(F; \mathcal{A}; \gamma)$ has the vertex $(0, \gamma/\nu(z))$ which does not correspond to any level.

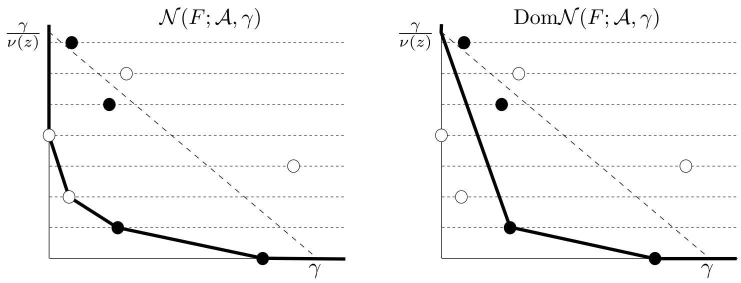

Figure 5.2: The Truncated Newton Polygon and the Dominant Truncated Newton Polygon

For each $k \geq 0$ let us consider the real number

$$\tau_k(F; \mathcal{A}; \gamma) = \min\{u \mid (u, k) \in \mathrm{Dom}\,\mathcal{N}(F; \mathcal{A}; \gamma)\} \ .$$

Note that $0 \leq \tau_k(F; \mathcal{A}; \gamma) \leq \max\{0, \gamma - k\nu(z)\}$.

Definition 18. A function $F \in R_{\mathcal{A}}^{\ell+1}$ is *γ-prepared in $\mathcal{A}$* if for any $0 \leq k \leq \gamma/\nu(z)$, the level F_k is $\tau_k(F; \mathcal{A}; \gamma)$-final.

The example represented in Figures 5.1 and 5.2 corresponds to a non γ-prepared function.

Proposition 7. *Given a function $F \in R_{\mathcal{A}}^{\ell+1}$ there is a ℓ-nested transformation $\mathcal{A} \to \mathcal{B}$ such that F is γ-prepared in $\mathcal{B}$.*

Proof. Let h be the integer part of $\gamma/\nu(z)$. By $T_4(\ell)$ there is a ℓ-nested transformation $\mathcal{A} \to \mathcal{A}_1$ such that F_0 is γ-final dominant in $\mathcal{A}_1$. In the same way, there is a ℓ-nested transformation $\mathcal{A}_1 \to \mathcal{A}_2$ such that F_1 is $(\gamma - \nu(z))$-final dominant in $\mathcal{A}_2$. By Lemma 5 we know that F_0 is γ-final dominant in $\mathcal{A}_2$. After performing a finite number of ℓ-nested transformation we obtain a parameterized regular local model $\mathcal{A}^*$ in which F_t is $(\gamma - t\nu(z))$-final for $t = 0, 1, \ldots, h$. Finally, performing the ℓ-nested transformation $\Psi_\ell : \mathcal{A}^* \to \mathcal{B}$ given by Lemma 8, all the levels F_k with $k > h$ becomes 0-final. □

A ℓ-nested transformation $\mathcal{A} \to \mathcal{B}$ such that F is γ-prepared in $\mathcal{B}$ is a *γ-preparation for F*.

Remark 17. Note that thanks to Lemma 5 given a γ-preparation for F

$$\mathcal{A} \to \mathcal{B}$$

and any ℓ-nested transformation

$$\mathcal{B} \to \mathcal{C}$$

then the composition

$$\mathcal{A} \to \mathcal{C}$$

of both ℓ-nested transformations is also a γ-preparation for F.

Note that if $F \in R_{\mathcal{A}}^{\ell+1}$ is γ-prepared then we have that $\mathcal{N}(F; \mathcal{A}; \gamma) = \mathrm{Dom}\,\mathcal{N}(F; \mathcal{A}; \gamma)$.

5.2 The critical height of a γ-prepared function

Let $F \in R_{\mathcal{A}}^{\ell+1}$ be a γ-prepared function. Recall that in this situation we have that $\mathcal{N}(F; \mathcal{A}; \gamma) = \mathrm{Dom}\,\mathcal{N}(F; \mathcal{A}; \gamma)$.

The *critical value* $\delta(F; \mathcal{A}; \gamma)$ is defined by

$$\delta(F; \mathcal{A}; \gamma) := \min_{k \geq 0} \{\tau_k(F; \mathcal{A}; \gamma) + k\nu(z)\} \ .$$

Note that $\delta(F; \mathcal{A}; \gamma) \leq \gamma$ since $(0, \gamma) \in \mathcal{N}(F; \mathcal{A}; \gamma)$. The critical value can be determined graphically:

$$\delta(F; \mathcal{A}; \gamma) = \min \left\{ \alpha \in \Gamma \ | \mathcal{N}(F; \mathcal{A}; \gamma) \cap L_{\nu(z)}(\alpha) \neq \emptyset \right\}$$

where $L_{\nu(z)}(\alpha)$ stands for the line passing though the point $(\alpha, 0)$ with slope $-1/\nu(z)$. If no confusion arises we denote the critical value by δ.

In the case $\delta < \gamma$ we say that $\mathcal{N}(F; \mathcal{A}; \gamma) \cap L_{\nu(z)}(\delta)$ is *the critical segment of* $\mathcal{N}(F; \mathcal{A}; \gamma)$. The highest vertex of the critical segment is the *critical vertex.* The height of the critical vertex is the *critical height of* $\mathcal{N}(F; \mathcal{A}; \gamma)$ and is denoted by $\chi(F; \mathcal{A}; \gamma)$. This integer number is our main control invariant. It satisfies

$$0 \leq \chi(F; \mathcal{A}; \gamma) \leq \frac{\delta}{\nu(z)} < \frac{\gamma}{\nu(z)} \ .$$

If no confusion arises we denote the critical height by χ.

Note that if $\delta(F; \mathcal{A}; \gamma) < \gamma$ we have

$$\delta(F; \mathcal{A}; \gamma) \geq \nu_{\mathcal{A}}(F) + \chi(F; \mathcal{A}; \gamma)\nu(z) \ ,$$

where we have equality if and only if $\nu_{\mathcal{A}}(F)$ is the abscissa of the critical vertex.

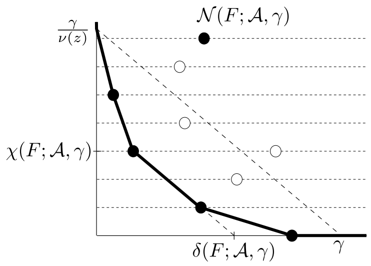

Figure 5.3: The critical value and the critical height

5.3 Pre-γ-final functions

Definition 19. A γ-prepared function $F \in R_{\mathcal{A}}^{\ell+1}$ is *pre-γ-final* if

$$\delta(F;\mathcal{A};\gamma) = \gamma$$

or

$$\delta(F;\mathcal{A};\gamma) < \gamma \quad \text{and} \quad \chi(F;\mathcal{A};\gamma) = 0 \ .$$

Pre-γ-final functions are easily recognizable by its Truncated Newton Polygon as it is represented in Figure 5.4 Let $\Psi_{\ell+1}$ be the $(\ell+1)$-nested transfor-

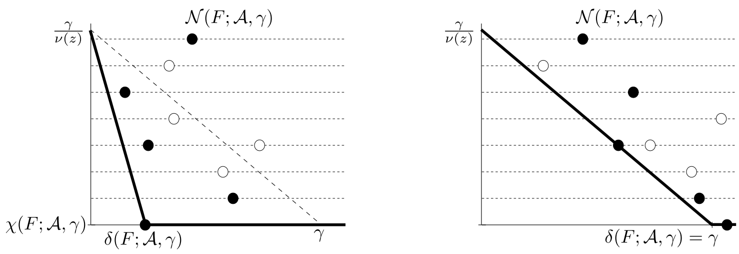

Figure 5.4: The two pre-γ-final situations

mation given in Lemma 8.

Proposition 8. *Let $F \in R_{\mathcal{A}}^{\ell+1}$ be a pre-γ-final function. Consider the $(\ell+1)$-nested transformation*

$$\mathcal{A} \xrightarrow{\pi} \mathcal{B} \xrightarrow{\Psi_{\ell+1}} \mathcal{C}$$

where $\pi : \mathcal{A} \to \mathcal{B}$ is a $(\ell+1)$-Puiseux's package. Then F is γ-final in $\mathcal{C}$.

Proof. First, suppose we have $\delta = \gamma$. In this situation for each index $k \geq 0$ we have

$$\nu_{\mathcal{A}}(F_k) \geq \gamma - k\nu(z) \ .$$

Let $\boldsymbol{x}'$ and z' be the variables in the parameterized regular local model $\mathcal{B}$ obtained after perform the $(\ell+1)$-Puiseux's package. From Equations (2.3) we know that

$$z = \boldsymbol{x}'^{\boldsymbol{\alpha}_0}(z'+\xi)^{\beta_0} \ , \quad \text{with } \nu(\boldsymbol{x}'^{\boldsymbol{\alpha}_0}) = \nu(z) \ ,$$

hence

$$\nu_{\mathcal{B}}(z^k) = \nu_{\mathcal{B}}\left(\boldsymbol{x}'^{k\boldsymbol{\alpha}_0}(z'+\xi)^{k\beta_0}\right) = k\nu(z) \ .$$

Therefore, for each $k \geq 0$ we have

$$\nu_{\mathcal{B}}(z^k F_k) = \nu_{\mathcal{B}}(z^k) + \nu_{\mathcal{B}}(F_k) \geq \gamma \ .$$

It follows that

$$\nu_{\mathcal{B}}(F) \geq \gamma \ .$$

If $\nu_{\mathcal{B}}(F) > \gamma$ then F is γ-final recessive in $\mathcal{B}$, so the same holds in $\mathcal{C}$ (Lemma 5). On the other hand, if $\nu_{\mathcal{B}}(F) = \gamma$, then F is γ-final (dominant or recessive) in $\mathcal{C}$ (see Lemma 8).

Now, suppose $\delta < \gamma$ and $\chi = 0$. In this situation for each index $k \geq 1$ we have

$$\nu_{\mathcal{A}}(F_k) > \delta - k\nu(z) \ ,$$

hence

$$\nu_{\mathcal{B}}(z^k F_k) > \delta \ .$$

As a consequence we have that

$$\nu_{\mathcal{B}}(F - F_0) = \nu_{\mathcal{B}}(F - \sum_{k \geq 1} F_k) > \delta \ ,$$

and therefore

$$\nu_{\mathcal{C}}(F - F_0) > \delta \ .$$

On the other hand we have that F_0 is dominant in $\mathcal{A}$ with explicit value δ, hence

$$\nu_{\mathcal{B}}(F_0) = \nu_{\mathcal{A}}(F_0) = \delta \ .$$

By Lemma 8 we know that F_0 is γ-final dominant in $\mathcal{C}$ with value δ. These facts imply that F is also γ-final dominant in $\mathcal{C}$ with value δ (note that $F = F_0 + (F - F_0)$). $\square$

5.4 Getting γ-final functions

In this section we will complete the proof of $T_4(\ell+1)$ by reductio ad absurdum: we suppose that we have a function $F \in R_{\mathcal{A}}^{\ell+1}$ such that there is no $(\ell+1)$-nested transformation $\mathcal{A} \to \mathcal{B}$ such that F is pre-γ-final in $\mathcal{B}$ and we will get a contradiction.

Let $\mathcal{A}$ be a parameterized regular local model and let $F \in R_{\mathcal{A}}^{\ell+1}$ be a function. Assume

1. F is γ-prepared;
2. for any $(\ell+1)$-nested transformation $\mathcal{A} \to \mathcal{B}$ we have that F is not pre-γ-final in $\mathcal{B}$.

For each index $k \geq 0$ let us write

$$F_k = \boldsymbol{x}^{\boldsymbol{q}_k} \tilde{F}_k + \bar{F}_k \ , \quad \nu_{\mathcal{A}}(\bar{F}_k) > \nu(\boldsymbol{x}^{\boldsymbol{q}_k}) \ ,$$

where we require $\tilde{F}_k \in k[[y_1, y_2, \ldots, y_\ell]]$. We have

$$\nu_{\mathcal{A}}(F_\chi) = \nu(\boldsymbol{x}^{\boldsymbol{q}_\chi}) = \delta - \chi\nu(z) \ .$$

Let $\phi \in K$ be the $(\ell+1)$-contact rational function $\phi_{\ell+1} = z^d/\boldsymbol{x}^{\boldsymbol{p}}$, where $d = d(\ell+1;\mathcal{A})$ is the ramification index (see Section 2.2.3).

Now, consider a level F_k which gives a point in the critical segment. We have that

$$\nu_{\mathcal{A}}(F_k) = \nu(\boldsymbol{x}^{\boldsymbol{q}_k}) = \delta - k\nu(z) = \nu(\boldsymbol{x}^{\boldsymbol{q}_\chi}) + (\chi - k)\nu(z) \ .$$

Therefore, the index k must be of the form $k = \chi - td$ for some integer $0 \leq t \leq \chi/d$. Following Remark 7, we have that

$$\boldsymbol{x}^{\boldsymbol{q}_{\chi-td}} = \boldsymbol{x}^{\boldsymbol{q}_\chi + t\boldsymbol{p}}$$

hence

$$z^{\chi-td}\tilde{F}_{\chi-td} = \boldsymbol{x}^{\boldsymbol{q}_\chi} z^\chi \phi^{-t} \tilde{F}_{\chi-td} \ .$$

Denote by M the integer part of χ/d. For $0 \leq t \leq M$ define the functions $G_t \in k[[y_1, y_2, \ldots, y_\ell]]$ given by

$$G_t = \begin{cases} \tilde{F}_{\chi-td} & \text{if } F_{\chi-td} \text{ gives a point in the critical segment;} \\ 0 & \text{otherwise .} \end{cases}$$

Note that if $G_t \neq 0$ then it is a unit. For $t = 0, 1, \ldots, M$ write

$$G_t = \tilde{G}_t + \bar{G}_t \ , \quad \tilde{G}_t \in k \ , \bar{G}_t \in \mathfrak{m}k[[y_1, y_2, \ldots, y_\ell]] \ .$$

Let $\tilde{F}_{\text{crit}}$ and $\bar{F}_{\text{crit}}$ be the functions given by

$$\tilde{F}_{\text{crit}} := \boldsymbol{x}^{\boldsymbol{q}_\chi} z^\chi \sum_{t=0}^{M} \phi^{-t}\tilde{G}_t \quad \text{and} \quad \bar{F}_{\text{crit}} := \boldsymbol{x}^{\boldsymbol{q}_\chi} z^\chi \sum_{t=0}^{M} \phi^{-t}\bar{G}_t \ .$$

Note that we have

$$\bar{F}_{\text{crit}} \in (y_1, y_2, \ldots, y_\ell) R_{\mathcal{A}}^{\ell+1} \ . \tag{5.1}$$

Denote by $\breve{F}$ the function

$$\breve{F} := F - \tilde{F}_{\text{crit}} - \bar{F}_{\text{crit}} \ .$$

Now we will study the behavior of F after performing a $(\ell+1)$-nested transformation of the kind

$$\mathcal{A} \xrightarrow{\pi} \tilde{\mathcal{B}} \xrightarrow{\tau} \mathcal{B}$$

where $\pi : \mathcal{A} \to \tilde{\mathcal{B}}$ is a $(\ell+1)$-Puiseux's package and $\tau : \tilde{\mathcal{B}} \to \mathcal{B}$ is a γ-preparation.

Perform a $(\ell+1)$-Puiseux's package $\mathcal{A} \to \tilde{\mathcal{B}}$ and let $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{z})$ be the coordinates in the parameterized regular local model $\tilde{\mathcal{B}}$. We have

$$\tilde{F}_{\text{crit}} = \tilde{\boldsymbol{x}}^{\boldsymbol{r}} \phi^e \sum_{t=0}^{M} \phi^{-t}\tilde{G}_t \quad \text{and} \quad \bar{F}_{\text{crit}} = \tilde{\boldsymbol{x}}^{\boldsymbol{r}} \phi^e \sum_{t=0}^{M} \phi^{-t}\bar{G}_t \ ,$$

where $\nu(\tilde{\boldsymbol{x}}^{\boldsymbol{r}}) = \delta$. The exponents $\boldsymbol{r} \in \mathbb{Z}_{\geq 0}^r$ and $e \in \mathbb{Z}_{>0}$ can be determined using the equalities given in (2.3). Recall that $\phi = \tilde{z} + \xi$ is a unit in $R_{\tilde{\mathcal{B}}}^{\ell+1}$. We can rewrite the above expressions as

$$\tilde{F}_{\text{crit}} = \tilde{\boldsymbol{x}}^{\boldsymbol{r}} U \sum_{t=0}^{M} (\tilde{z}+\xi)^{M-t}\tilde{G}_t \quad \text{and} \quad \bar{F}_{\text{crit}} = \tilde{\boldsymbol{x}}^{\boldsymbol{r}} U \sum_{t=0}^{M} (\tilde{z}+\xi)^{M-t}\bar{G}_t \ ,$$

where $U = \phi^{e-M}$ is a unit. Note that we have

$$\nu_{\tilde{\mathcal{B}}}(\tilde{F}_{\text{crit}}) = \nu_{\tilde{\mathcal{B}}}(\bar{F}_{\text{crit}}) = \delta \ . \tag{5.2}$$

On the other hand, it follows by construction that

$$\nu_{\tilde{\mathcal{B}}}(\breve{F}) > \delta \ . \tag{5.3}$$

Note that Equation (5.1) gives

$$\bar{F}_{\text{crit}} \in (\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_\ell) R^{\ell+1}_{\tilde{\mathcal{A}}} \ . \tag{5.4}$$

Let $Q \in k[\tilde{z}]$ be the polynomial

$$Q = \sum_{t=0}^{M} \tilde{G}_t (\tilde{z} + \xi)^{M-t} \ ,$$

and denote by $\hbar$ its order. Note that $\hbar \leq M \leq \chi$ and

$$\begin{aligned} \hbar = M \quad &\Longleftrightarrow \quad Q = \tilde{G}_0 \tilde{z}^M \\ &\Longleftrightarrow \quad \tilde{G}_t = (-1)^t \xi^t \binom{\chi}{t} \mu_0 \quad \text{for } 1 \leq t \leq M \ . \end{aligned} \tag{5.5}$$

From Equations (5.2), (5.3) and (5.1) we have that the $\hbar$-level of F in $\tilde{\mathcal{B}}$ is dominant.

Now, perform a γ-preparation $\tilde{\mathcal{B}} \to \mathcal{B}$. Let δ' be the critical value of F in $\mathcal{B}$. By assumption we have $\delta' < \gamma$. Let χ' be the new critical height.

Since the $\hbar$-level of F in $\tilde{\mathcal{B}}$ is dominant, the same happens in $\mathcal{B}$ (Lemma 5). We also have that

$$\nu_{\mathcal{B}}(F) = \nu_{\tilde{\mathcal{B}}}(F) = \nu_{\tilde{\mathcal{B}}}(\tilde{F}_{\text{crit}} + \bar{F}_{\text{crit}} + \breve{F}) = \delta \ .$$

We conclude that

$$\chi' \leq \hbar \leq M = \left[\frac{\chi}{d}\right] \leq \chi \ . \tag{5.6}$$

Inequality (5.6) gives

$$\chi' < \chi$$

except in the case when $d = 1$ and the condition about the coefficients of $\tilde{F}_{\text{crit}}$ given in (5.5) is satisfied. Note that we have

$$\chi = \chi' \Longrightarrow \nu_{\mathcal{B}}(F) = \delta = \delta' - \chi \nu(z') \ , \tag{5.7}$$

where z' is the $(\ell+1)$-th dependent variable in $\mathcal{B}$.

Suppose that $\chi' = \chi$. In this situation instead of performing the $(\ell+1)$-nested transformation

$$\mathcal{A} \xrightarrow{\pi} \tilde{\mathcal{B}} \xrightarrow{\tau} \mathcal{B}$$

we will make an ordered change of the variable z.

So we have a parameterized regular local model $\mathcal{A}$ with $d = d(\ell+1; \mathcal{A}) = 1$ and a function $F \in R^{\ell+1}_{\mathcal{A}}$ with critical height χ and such that the coefficients of $\tilde{F}_{\text{crit}}$ satisfy the condition given in (5.5). Furthermore, following Equation (5.7), after performing a $(\ell+1)$-Puiseux's package and a γ-preparation if necessary, we can assume that

$$\nu_{\mathcal{A}}(F) = \nu_{\mathcal{A}}(F_\chi) \ .$$

Moreover, after performing a 0-nested transformation given by Lemma 4 if necessary, we can suppose that F_χ divides F. So, after factoring F_χ, we can assume that

$$\nu_{\mathcal{A}}(F) = \nu_{\mathcal{A}}(F_\chi) = 0 \quad \text{and} \quad F_\chi = 1 \ .$$

Since F is γ-prepared, the level at height $(\chi-1)$ has the form

$$F_{\chi-1} = \boldsymbol{x}^{\boldsymbol{p}} \tilde{F}_{\chi-1} + \bar{F}_{\chi-1} \ , \quad \nu_{\mathcal{A}}(\bar{F}_{\chi-1}) > \nu(\boldsymbol{x}^{\boldsymbol{p}}) \ ,$$

where $\tilde{F}_{\chi-1}$ is a unit which does not depend on the independent variables $\boldsymbol{x}$. Let us write $\tilde{F}_{\chi-1}$ as a power series

$$\tilde{F}_{\chi-1} = \sum_{(I,J)\in\mathbb{Z}_{\geq 0}^{r+\ell}} f_{IJ} \boldsymbol{x}^I \boldsymbol{y}^J \ , \quad f_{IJ} \in k \ .$$

Denote

$$\tilde{F}_{\chi-1} = G + H$$

where $G \in k[\boldsymbol{x}, \boldsymbol{y}] \subset R^{\ell}_{\mathcal{A}}$ is the polynomial

$$G = \sum_{\substack{(I,J)\in\mathbb{Z}_{\geq 0}^{r+\ell} \\ \nu(\boldsymbol{x}^I \boldsymbol{y}^J) \leq 2\nu(z)}} f_{IJ} \boldsymbol{x}^I \boldsymbol{y}^J \ .$$

Since the coefficients of F satisfy the conditions given in (5.5) we have

$$G = -\xi\chi \boldsymbol{x}^{\boldsymbol{p}} + \cdots \ .$$

Note that

$$\nu_{\mathcal{A}}(\tilde{F}_{\chi-1}) = \nu_{\mathcal{A}}(G) = \nu(z) \leq \nu_{\mathcal{A}}(H) \ . \tag{5.8}$$

Now consider the ordered change of coordinates

$$\tilde{z} = z - \phi \ , \quad \text{where} \quad \phi := \frac{-1}{\chi} G \ ,$$

and let $\tilde{\mathcal{A}}$ be the parameterized regular local model obtained. Note that

$$\nu(\tilde{z}) \geq \nu(z) \ .$$

We have

$$F = \sum_{k=0}^{\infty} z^k F_k = \sum_{k=0}^{\infty} (\tilde{z} + \phi)^k F_k = \sum_{k=0}^{\infty} \tilde{z}^k F'_k \ ,$$

where

$$F'_k = F_k + \sum_{i=1}^{\infty} \binom{k+i}{i} \phi^i F_{k+i} \ .$$

So the $(\chi-1)$-level of F in $\tilde{\mathcal{A}}$ is

$$F'_{\chi-1} = F_{\chi-1} + \chi\phi F_\chi + \phi^2(\cdots) = G + H - G + \phi^2(\cdots) = H + \phi^2(\cdots) \ . \tag{5.9}$$

In $\tilde{\mathcal{A}}$ the function F is not necessarily γ-prepared so let $\tilde{\mathcal{A}} \to \mathcal{A}_1$ a γ-preparation.

It follows from the definition of H and Equations (5.8) and (5.9) that

$$\nu_{\mathcal{A}_1}(F'_{\chi-1}) \geq 2\nu(z) \ .$$

Note also that we still have

$$\nu_{\mathcal{A}_1}(F) = \nu_{\mathcal{A}_1}(F'_{\chi}) = 0 \ .$$

Let z_1 be the $(\ell+1)$-th dependent variable in $\mathcal{A}_1$. We have that

$$\chi(F; \mathcal{A}_1, \gamma) \leq \chi(F; \mathcal{A}; \gamma) \ .$$

Furthermore, we have

$$\chi(F; \mathcal{A}_1, \gamma) = \chi(F; \mathcal{A}; \gamma) \Longrightarrow \nu(z_1) = \nu_{\mathcal{A}_1}(F'_{\chi-1}) \geq 2\nu(z) \ . \tag{5.10}$$

Now, we can perform a z_1-Puiseux's package. If the critical height does not drop, instead of performing a z_1-Puiseux's package we make an ordered change of coordinates as above. We iterate this procedure while the critical height does not drop. At each step we obtain a parameterized regular local model $\mathcal{A}_i$. By Equation (5.10) we know that the $(\ell+1)$-th dependent variable z_i satisfies

$$\nu(z_i) \geq 2^i \nu(z) \ .$$

This can not happen infinitely many times, since in a finite number of steps we reach a parameterized regular local model $\mathcal{A}_{i_0}$ such that

$$\nu(z_{i_0}) \geq \frac{\gamma - \nu_{\mathcal{A}}(F)}{\chi} = \frac{\gamma - \nu_{\mathcal{A}_{i_0}}(F)}{\chi} \ .$$

The above inequality implies that $\delta(F; \mathcal{A}_{i_0}, \gamma) = \gamma$ which is in contradiction with our assumptions.

Then, after a finite number of ordered changes of the $(\ell+1)$-th variable and γ-preparations of F, necessarily we reach a parameterized regular local model in which the critical height drops by means of a $(\ell+1)$-Puiseux's package. Again, this can not happen infinitely many times since we are assuming that the critical height is strictly positive.

We have just proved that our assumptions give a contradiction, so there is always a $(\ell+1)$-nested transformation which transform a function into a γ-final one. Thus, we have prove that

$$T_4(\ell) \Longrightarrow T_4(\ell+1) \ .$$

Chapter 6

Truncated preparation of a 1-form

In this chapter and the next one we will detail the proof of

$$T_3(\ell) \Longrightarrow T_3(\ell+1) \ .$$

We will adapt the arguments used in Chapter 5 to the case of 1-forms. As the name of the chapter indicates, this chapter is the equivalent for 1-forms of the Section 5.1.

Let $\mathcal{A} = \big(\mathcal{O}, (\boldsymbol{x}, \boldsymbol{y})\big)$ be a parameterized regular local model of K, ν. Fix an index ℓ, $0 \leq \ell \leq n-r-1$ and a value $\gamma \in \Gamma$. We consider a 1-form $\omega \in N_{\mathcal{A}}^{l+1}$ such that $\nu_{\mathcal{A}}(\omega \wedge d\omega) \geq 2\gamma$. Since we are working by induction on ℓ, we assume that the statement $T_3(\ell)$ is true (hence $T_4(\ell)$ and $T_5(\ell)$ are also true).

6.1 Expansions relative to a dependent variable

Let us denote the dependent variable y_{l+1} by z. Note that by definition

$$R_{\mathcal{A}}^{l+1} = R_{\mathcal{A}}^{\ell}[[z]] \ .$$

Thus we can expand an element $F \in R_{\mathcal{A}}^{l+1}$ as a power series in the variable z:

$$F = \sum_{k=0}^{\infty} F_k z^k \quad , \ F_k \in R_{\mathcal{A}}^{\ell}.$$

Take an element of $N_{\mathcal{A}}^{l+1}$

$$\omega = \sum_{i=1}^{r} a_i \frac{dx_i}{x_i} + \sum_{j=1}^{\ell} b_j dy_j + c dz \ .$$

Write

$$\omega = \sum_{i=1}^{r} a_i \frac{dx_i}{x_i} + \sum_{j=1}^{\ell} b_j dy_j + f \frac{dz}{z} \ ,$$

where $f = zc$. The *decomposition in z-levels of* ω of consists in writing ω as

$$\omega = \sum_{k=0}^{\infty} z^k \omega_k = \sum_{k=0}^{\infty} z^k \Big(\sum_{i=1}^{r} a_{ik} \frac{dx_i}{x_i} + \sum_{j=1}^{\ell} b_{jk} dy_j + f_k \frac{dz}{z} \Big) \ . \tag{6.1}$$

where

$$a_i = \sum_{k=0}^{\infty} a_{ik} z^k \ , \ b_j = \sum_{k=0}^{\infty} b_{jk} z^k \ \text{ and } \ f = \sum_{k=1}^{\infty} f_0 z^k \ .$$

Note that $f_0 = 0$. We say that ω_k is *the k-level of* ω.

Remark 18. The coefficients of each z-level ω_k are elements of $R^{\ell}_{\mathcal{A}}$, but ω_k itself belongs neither to $N^{\ell}_{\mathcal{A}}$ nor to $N^{l+1}_{\mathcal{A}}$. The z-levels ω_k belong to the $R^{\ell}_{\mathcal{A}}$-module $N^{\ell}_{\mathcal{A}} \oplus R^{\ell}_{\mathcal{A}} \frac{dz}{z}$. We will write

$$\omega_k = \eta_k + f_k \frac{dz}{z} \in N^{\ell}_{\mathcal{A}} \oplus R^{\ell}_{\mathcal{A}} \frac{dz}{z} \ , \ \forall k \geq 0 \ ,$$

where we denote by $\eta_k \in N^{\ell}_{\mathcal{A}}$ the 1-forms

$$\eta_k := \sum_{i=1}^{r} a_{ik} \frac{dx_i}{x_i} + \sum_{j=1}^{\ell} b_{jk} dy_j \ , \ \forall k \geq 0 \ .$$

To each level we can attach a pair

$$\omega_k = \eta_k + f_k \frac{dz}{z} \longmapsto (\eta_k, f_k) \in N^{\ell}_{\mathcal{A}} \times R^{\ell}_{\mathcal{A}} \ .$$

Denote by $\delta_k(\omega; \mathcal{A}), \phi_k(\omega; \mathcal{A}), \beta_k(\omega; \mathcal{A}) \in \Gamma \cup \{\infty\}$ the explicit values

$$\begin{aligned} \delta_k(\omega; \mathcal{A}) &:= \nu_{\mathcal{A}}(\eta_k) \ , \\ \phi_k(\omega; \mathcal{A}) &:= \nu_{\mathcal{A}}(f_k) \ , \\ \beta_k(\omega; \mathcal{A}) &:= \nu_{\mathcal{A}}(\eta_k, f_k) = \min \{\phi_k(\omega; \mathcal{A}), \eta_k(\omega; \mathcal{A})\} \ . \end{aligned}$$

The value $\beta_k(\omega; \mathcal{A})$ is the *explicit value of* ω_k. If no confusion arises we denote $\delta_k(\omega; \mathcal{A})$, $\phi_k(\omega; \mathcal{A})$ and $\beta_k(\omega; \mathcal{A})$ by δ_k, ϕ_k and β_k respectively. Given $\alpha \in \Gamma$ we say that the level ω_k is *α-final* (*final dominant*, *final recessive*) if and only if the pair (η_k, f_k) is α-final (final dominant, final recessive). In particular, we say that a level ω_k is *log-elementary* if it is 0-final dominant, and that it is *dominant* if it is β_k-final dominant. Write

$$\omega_k = \boldsymbol{x}^{\boldsymbol{t}} \left(\tilde{\eta}_k + \tilde{f}_k \frac{dz}{z} \right) + \bar{\eta}_k + \bar{f}_k \frac{dz}{z}$$

where $\tilde{f}_k$ and the coefficients of $\tilde{\eta}_k$ belong to $k[[\boldsymbol{y}]]$ and

$$\nu_{\mathcal{A}}(\omega_k) = \nu(\boldsymbol{x}^{\boldsymbol{t}}) \quad \text{and} \quad \nu_{\mathcal{A}}(\bar{\eta}_k, \bar{f}_k) > \nu(\boldsymbol{x}^{\boldsymbol{t}}) \ .$$

We define the *$\mathcal{A}$-initial part of* ω_k as

$$\mathrm{in}_{\mathcal{A}}(\omega_k) := \boldsymbol{x}^{\boldsymbol{t}} \left(\mathrm{in}_{\mathcal{A}}(\tilde{\omega}_k) + \tilde{f}_k(\mathbf{0}) \frac{dz}{z} \right) \ , \tag{6.2}$$

where we recall that the $\mathcal{A}$-initial part of $\tilde{\omega}_k \in N^\ell_{\mathcal{A}}$ was defined in Section 4.1. As in the case of elements of $N^\ell_{\mathcal{A}}$ (see Remark 13) we have that a level $\omega_k \in N^\ell_{\mathcal{A}} \oplus R^\ell_{\mathcal{A}} \frac{dz}{z}$ is $\nu_{\mathcal{A}}(\omega_k)$-final dominant if and only if $\mathrm{in}_{\mathcal{A}}(\omega_k) \neq 0$.

From Lemmas 5, 6 and 7 we obtain the following property of stability of δ_k, ϕ_k and β_k under any ℓ-nested transformation:

> **Property of stability of levels**. For any ℓ-nested transformation $\mathcal{A} \to \mathcal{A}'$ and any $k \geq 0$, we have that $\delta'_k \geq \delta_k$, $\phi'_k \geq \phi_k$ and $\beta'_k \geq \beta_k$.

In addition we have stability for dominant levels:

> **Property of stability for dominant levels**. Given a dominant level ω_k and any ℓ-nested transformation $\mathcal{A} \to \mathcal{A}'$, the transformed level ω'_k is also dominant and $\beta'_k = \beta_k$.

6.2 Truncated Newton polygons and prepared 1-forms

Using the values defined in the previous section we define certain subsets of $\Gamma_{\geq 0} \times \mathbb{Z}_{\geq 0} \subset \mathbb{R}^2_{\geq 0}$. The *Cloud of Points of* ω is the discrete subset

$$\mathrm{CL}(\omega; \mathcal{A}) := \{(\beta_k, k) \mid k = 0, 1, \dots\} \ .$$

Note that

$$\omega \neq 0 \Rightarrow \mathrm{CL}(\omega; \mathcal{A}) \neq \emptyset \ .$$

We also use the *Dominant Cloud of Points of* ω

$$\mathrm{DomCL}(\omega; \mathcal{A}) := \{(\beta_k, k) \in \mathrm{CL}(\omega; \mathcal{A}) \mid \omega_k \text{ is dominant}\} \ .$$

Note that $\mathrm{DomCL}(\omega; \mathcal{A})$ can be empty. In Figure 6.1 we can see an example in which the points (β_k, k) are represented with black and white circles, corresponding to dominant and non-dominant levels respectively.

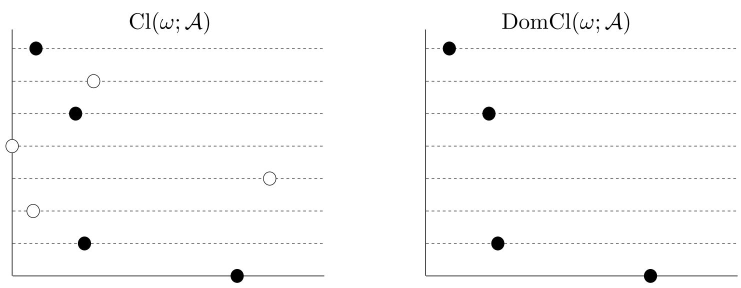

Figure 6.1: The Cloud of Points and the Dominant Cloud of Points

Given a value $\sigma \in \Gamma$, we define the truncated polygons

$$\mathcal{N}(\omega; \mathcal{A}, \sigma) \quad \text{and} \quad \mathrm{Dom}\mathcal{N}(\omega; \mathcal{A}, \sigma)$$

to be the respective positively convex hulls of

$$\{(0, \sigma/\nu(z)), (\sigma, 0)\} \cup \mathrm{CL}(\omega; \mathcal{A})$$

and

$$\{(0,\sigma/\nu(z)),(\sigma,0)\} \cup \mathrm{DomCL}(\omega;\mathcal{A}) \ .$$

Note that for any $\sigma \in \Gamma$ we have that

$$\mathcal{N}(\omega;\mathcal{A},\sigma) \supset \mathrm{Dom}\,\mathcal{N}(\omega;\mathcal{A},\sigma) \ .$$

In Figure 6.2 we can see the truncated polygons corresponding to the cloud of points represented in Figure 6.1. Note that in this example $\mathrm{Dom}\,\mathcal{N}(\omega;\mathcal{A};\gamma)$ has the vertex $(0,\gamma/\nu(z))$ which does not correspond to any level.

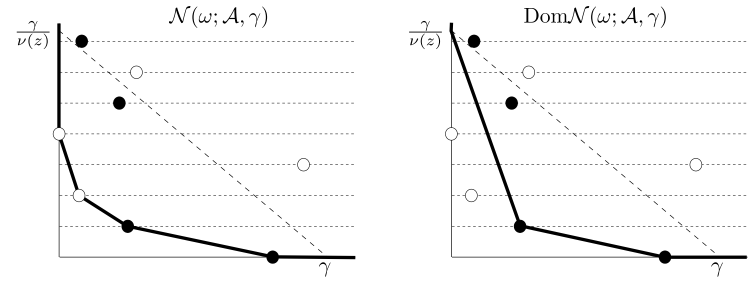

Figure 6.2: The Truncated Newton Polygon and the Dominant Truncated Newton Polygon

For each $k \geq 0$ let us consider the real number

$$\tau_k(\omega;\mathcal{A};\gamma) := \min\{u \mid (u,k) \in \mathrm{Dom}\,\mathcal{N}(\omega;\mathcal{A};\gamma)\} \ .$$

Note that $0 \leq \tau_k(\omega;\mathcal{A};\gamma) \leq \max\{0, \gamma - k\nu(z)\}$.

Definition 20. We say that ω is *γ-prepared* in $\mathcal{A}$ if the level ω_k is $\tau_k(\omega;\mathcal{A};\gamma)$-final for any $0 \leq k \leq \gamma/\nu(z)$.

The example represented in Figures 6.1 and 6.2 corresponds to a non γ-prepared 1-form.

Remark 19. Note that being γ-prepared implies that $\mathcal{N}(\omega;\mathcal{A};\gamma) = \mathrm{Dom}\mathcal{N}$. Conversely, if we have that $\mathcal{N}(\omega;\mathcal{A};\gamma) = \mathrm{Dom}\mathcal{N}$ to assure that ω is γ-prepared it is enough to guarantee that $\beta_k > \tau_k$ for any level ω_k which is not τ_k-dominant. This last condition can be obtained applying Lemma 8.

The objective of this chapter is to prove the following result

Theorem 6 (Existence of γ-preparation)**.** *Let $\omega \in N_{\mathcal{A}}^{\ell+1}$ be a 1-form such that $\nu_{\mathcal{A}}(\omega \wedge d\omega) \geq 2\gamma$. There is a ℓ-nested transformation $\mathcal{A} \to \mathcal{B}$ such that ω is γ-prepared in $\mathcal{B}$.*

A ℓ-nested transformation $\mathcal{A} \to \mathcal{B}$ such that ω is γ-prepared in $\mathcal{B}$ is called a *γ-preparation.*

6.3 Property of preparation of levels

Consider and integer number $k \geq 0$ and put

$$\lambda_k(\omega;\mathcal{A};\gamma) = \min\{\gamma - k\nu(z)\} \cup \left\{ \frac{\delta_{k+s}+\delta_{k-s}}{2} \mid \quad s \geq 1 \right\} \ . \tag{6.3}$$

In this section we prove the following Lemma

Lemma 9. *There is an ℓ-nested transformation $\mathcal{A} \to \mathcal{A}'$ such that the k-level ω'_k of ω with respect to $\mathcal{A}'$ is $\lambda_k(\mathcal{A}; \omega, \gamma)$-final.*

This result is a consequence of the induction hypothesis and the fact that $\nu_{\mathcal{A}}(\omega \wedge d\omega) \geq 2\gamma$. Namely, we can write

$$\omega \wedge d\omega = \sum_{m=0}^{\infty} z^m \Big(\Theta_m + \frac{dz}{z} \wedge \Delta_m \Big)$$

where

$$\Theta_m := \sum_{i+j=m} \eta_i \wedge d\eta_j$$

and

$$\Delta_m := \sum_{i+j=m} j\eta_j \wedge \eta_i + f_i d\eta_j + \eta_i \wedge df_j \ .$$

We have that

$$\nu_{\mathcal{A}}(\omega \wedge d\omega) \geq 2\gamma$$

is equivalent to

$$\nu_{\mathcal{A}}(\Theta_m) \geq 2\gamma \text{ and } \nu_{\mathcal{A}}(\Delta_m) \geq 2\gamma \quad \forall\, m \geq 0 \ .$$

The proof of Lemma 9 is based on this equivalence. In view of the statement $T_3(\ell)$ it is enough to prove that

$$\nu_{\mathcal{A}}(\eta_k \wedge d\eta_k) \geq 2\lambda_k \ .$$

Look at Θ_{2k}:

$$\Theta_{2k} = \eta_k \wedge d\eta_k + \sum_{\substack{i+j=2k \\ i,j\neq k}} \eta_i \wedge d\eta_j \ .$$

Recall that $\nu_{\mathcal{A}}(\Theta_{2k}) \geq 2\gamma$. By definition of the values δ_i, we have that $\nu_{\mathcal{A}}(\eta_i) \geq \delta_i$, hence $\nu_{\mathcal{A}}(d\eta_i) \geq \delta_i$. Writing

$$\eta_k \wedge d\eta_k = -\Theta_{2k} + \sum_{\substack{i+j=2k \\ i,j\neq k}} \eta_i \wedge d\eta_j \ ,$$

we conclude that $\nu_{\mathcal{A}}(\eta_k \wedge d\eta_k) \geq 2\lambda_k$. We end the proof of Lemma 9 applying $T_5(\ell)$ to the pair (η_k, f_k).

6.4 Preparation. First reductions

In order to prove Theorem 6 we will first show that we can assume some reductions.

Proposition 9. *Let $\omega \in N_{\mathcal{A}}^{\ell+1}$ be a 1-form such that $\nu_{\mathcal{A}}(\omega \wedge d\omega) \geq 2\gamma$. Without lost of generality we can assume that the following properties are satisfied:*

1. ***Maximality of dominant levels**: For any integer number k with $0 \leq k \leq \gamma/\nu(z)$, the level ω_k is either $(\gamma - k\nu(z))$-final dominant in $\mathcal{A}$ or there is no ℓ-nested transformation $\mathcal{A} \to \mathcal{A}'$ such that ω_k is $(\gamma - k\nu(z))$-final dominant in $\mathcal{A}'$;*

2. ***Preparation of the functional part:*** *For any integer number k with $0 \leq k \leq \gamma/\nu(z)$, the function $f_k \in R^{\ell}_{\mathcal{A}}$ is $(\gamma - k\nu(z))$-final;*

3. ***Preparation of the* 0*-level:*** *The 0-level ω_0 is γ-final;*

4. ***Explicitness of the dominant vertices:*** *Any vertex of $\mathrm{Dom}\,\mathcal{N}(\omega;\mathcal{A};\gamma)$ is also a vertex of $\mathcal{N}(\omega;\mathcal{A};\gamma)$.*

First of all, note that the four properties listed in the proposition are stable under further ℓ-nested transformations. The first property can be obtained without making use of the induction hypothesis while the remaining ones needs the assumption that $T_3(\ell)$ is true. In this section we detail how to obtain the first three properties. The following section is devoted to show how to obtain explicit dominant vertices.

Maximality of dominant levels: This property can be obtained thanks to the stability of dominant levels as follows. Take an integer k with $0 \leq k \leq \gamma/\nu(z)$. If there is a ℓ-nested transformation such that ω_k becomes $(\gamma - k\nu(z))$-final dominant, perform it. In this way we perform a finite number of transformations to get the desired maximality property.

Preparation of the functional part: We only have to use $T_4(\ell)$ finitely many times.

Preparation of the 0-level: This property is also obtained using the induction hypothesis. We have that $\nu_{\mathcal{A}}(\omega \wedge d\omega) \geq 2\gamma$ implies $\nu_{\mathcal{A}}(\Theta_0) \geq 2\gamma$. Since $\Theta_0 = \eta_0 \wedge d\eta_0$ we can invoke $T_3(\ell)$ and transform ω_0 into a γ-final level.

6.5 Getting explicit dominant vertices

In this section we complete the proof of Proposition 9 by showing how to obtain explicit dominant vertices. First of all, note that the maximality of dominant vertices property implies the following additional property:

> **Stability of the Truncated Dominant Newton Polygon**: For any ℓ-nested transformation $\mathcal{A} \to \mathcal{A}'$ we have that
> $$\mathrm{Dom}\mathcal{N}(\omega;\mathcal{A};\gamma) = \mathrm{Dom}\mathcal{N}(\omega;\mathcal{A}',\gamma)\ .$$

In view of this stability property, from now on we will denote the Truncated Dominant Newton Polygon $\mathrm{Dom}\mathcal{N}(\omega;\mathcal{A};\gamma)$ simply by $\mathrm{Dom}\mathcal{N}$, and the values $\tau_k(\omega;\mathcal{A};\gamma)$ by τ_k.

For any positive real number $\delta > 0$, let us consider the lines

$$L_\delta(\rho) = \{(u,v) \mid u + \delta v = \rho\}$$

of slope $-1/\delta$, and the open half-planes

$$H^+_\delta(\rho) = \{(u,v) \mid u + \delta v > \rho\} \quad \text{and} \quad H^-_\delta(\rho) = \{(u,v) \mid u + \delta v < \rho\}\ .$$

Let ρ_δ be the real number defined by

$$\rho_\delta := \min\{\rho \mid L_\delta(\rho) \cap \mathrm{Dom}\,\mathcal{N} \neq \emptyset\} = \sup\{\rho \mid H^+_\delta(\rho) \supset \mathrm{Dom}\,\mathcal{N}\}\ .$$

We have that $L_\delta(\rho_\delta)$ cuts the polygon $\mathrm{Dom}\,\mathcal{N}$ in only one vertex or a side joining two vertices.

In order to get explicit dominant vertices, it is enough to prove the following lemma:

Lemma 10. *Given $\delta > 0$ and a fixed $\epsilon > 0$ there is an ℓ-nested transformation $\mathcal{A} \to \mathcal{A}'$ such that $\mathcal{N}(\omega; \mathcal{A}', \gamma) \subset H^+_\delta(\rho_\delta - \epsilon)$.*

Note that as usual, the property obtained after applying the lemma is stable under new ℓ-nested transformations.

Let us show that Lemma 10 allows us to get the explicitness of dominant vertices property. Consider a vertex $\boldsymbol{v} = (\tau_k, k)$ of $\mathrm{Dom}\,\mathcal{N}$. Take two slopes $0 < \delta_2 < \delta_1$ such that both $L_{\delta_1}(\rho_{\delta_1})$ and $L_{\delta_2}(\rho_{\delta_2})$ cut the polygon $\mathrm{Dom}\,\mathcal{N}$ only in the vertex $\boldsymbol{v}$. Consider an slope δ_3 with $\delta_2 < \delta_3 < \delta_1$. We also have that $L_{\delta_3}(\rho_{\delta_3})$ cuts $\mathrm{Dom}\,\mathcal{N}$ only in the vertex $\boldsymbol{v}$. By an elementary geometrical argument, we see that there is an $\epsilon > 0$ satisfying the following property

> "For any $(a, k') \in H^+_{\delta_1}(\rho_{\delta_1} - \epsilon) \cap H^+_{\delta_2}(\rho_{\delta_2} - \epsilon)$ such that $k' \neq k$ we have that $(a, k') \in H^+_{\delta_3}(\rho_{\delta_3})$".

Applying Lemma 10 with respect to ϵ, we obtain that $\boldsymbol{v}$ is a vertex of $\mathcal{N}(\omega; \mathcal{A}', \gamma)$. Repeating this argument at all the vertices of $\mathrm{Dom}\,\mathcal{N}$ we obtain the explicitness of dominant vertices property.

So in order to complete the proof of Proposition 9 we have to prove Lemma 10. Denote by (α_k, k) the point in $\mathcal{N}(\omega; \mathcal{A}; \gamma)$ with integer ordinate equal to k and smallest abscissa. Note that

$$\alpha_k \leq \tau_k \ .$$

Note also that $\alpha_0 \leq \gamma$ and $\alpha_k = 0$ for any $k > \gamma/\nu(z)$. We use as a key argument the following property

> **Reduction of vertices**: Consider a vertex $\boldsymbol{v} = (\alpha_k, k)$ of the polygon $\mathcal{N}(\omega; \mathcal{A}; \gamma)$ which is not a vertex of $\mathrm{Dom}\,\mathcal{N}$ (in particular $k \geq 1$). There is a ℓ-nested transformation $\mathcal{A} \to \mathcal{A}$ such that
>
> $$\mathcal{N}(\omega; \mathcal{A}', \lambda) \subset \mathcal{N}'$$
>
> where $\mathcal{N}'$ is the positively convex polygon generated by $\{(\alpha'_s, s)\}_{s \geq 0}$ where
>
> $$\alpha'_s = \begin{cases} \alpha_s \ , & \text{if } s \neq k \ ; \\ \frac{\alpha_{k-1} + \alpha_{k+1}}{2} \ , & \text{if } s = k \ . \end{cases}$$

This property is a direct consequence of Lemma 9. Note that the k-level is never dominant, since $\boldsymbol{v}$ is a vertex of $\mathcal{N}(\omega; \mathcal{A}; \gamma)$ which is not a vertex of $\mathrm{Dom}\,\mathcal{N}$ and the property of maximality of dominant levels holds.

Now, take a positively convex polygon $\mathcal{N}$ of $\mathbb{R}^2_{\geq 0}$ such that all the vertices have integer ordinates, except maybe the vertex of abscissa 0. Consider δ and ρ_δ as in the statement of Lemma 10. Note that

$$\gamma/\nu(z) \geq \rho_\delta/\delta \quad \text{and} \quad \gamma \geq \rho_\delta \ .$$

Suppose also that

1. Either $(\rho_\delta, 0)$ is a vertex of $\mathcal{N}$ or $(\rho_\delta, 0) \notin \mathcal{N}$;

2. Either $(0, \rho_\delta/\delta)$ is a vertex of $\mathcal{N}$ or $(0, \rho_\delta/\delta) \notin \mathcal{N}$;

3. The points $(0, \gamma/\nu(z))$ and $(\gamma, 0)$ are in $\mathcal{N}$.

For any vertex $\boldsymbol{v}$ of $\mathcal{P}$ denote by $\delta_l(\boldsymbol{v})$ and by $\delta_r(\boldsymbol{v})$ the real numbers such that $-1/\delta_l(\boldsymbol{v})$ and $-1/\delta_r(\boldsymbol{v})$ are the slopes of the two sides of $\mathcal{N}$ throughout $\boldsymbol{v}$, with $0 \leq \delta_l(\boldsymbol{v}) < \delta_r(\boldsymbol{v}) \leq +\infty$. The following Lemma has an elementary proof:

Lemma 11. *In the above situation, there is a constant $K_\epsilon \geq 0$, not depending on the particular polygon, such that for any $\mathcal{N}$ with*

$$\mathcal{N} \not\subset H_\delta^+(\rho_\delta - \epsilon) \ ,$$

there is at least one vertex $\boldsymbol{v}$ of $\mathcal{N}$ with $\boldsymbol{v} \in \mathcal{N} \cap H_\delta^-(\rho_\delta)$ such that

$$\delta_\ell(\boldsymbol{v}) < \delta_r(\boldsymbol{v}) - K_\epsilon \ .$$

Proof. Take $k_\epsilon \in \mathbb{R}$ such that

$$k_\epsilon < \frac{2\delta^2\epsilon}{\rho_\delta(\rho_\delta + \delta)} \ .$$

We assert that this constant satisfies the conditions required in the lemma. Suppose that it is false. Consider a polygon $\mathcal{N} \not\subset H_\delta^+(\rho_\delta - \epsilon)$. Since $\mathcal{N}$ is not contained in $H_\delta^+(\rho_\delta - \epsilon)$ there must be a vertex $\boldsymbol{v}$ of $\mathcal{N}$ such that $\boldsymbol{v} \notin H_\delta^+(\rho_\delta - \epsilon)$. If our assumption is false, for every vertex $\boldsymbol{v}'$ in $H_\delta^-(\rho_\delta)$ (and in particular for $\boldsymbol{v}$) we have

$$\delta_\ell(\boldsymbol{v}') \geq \delta_r(\boldsymbol{v}') - K_\epsilon \ .$$

But this condition gives that at least one of the points $(0, \rho_\delta/\delta)$ or $(\rho_\delta, 0)$ are interior points of $\mathcal{N}$ in contradiction with the hypothesis about $\mathcal{N}$. □

Now, let us apply Lemma 11 to prove Lemma 10. Assume that

$$\mathcal{N}(\omega; \mathcal{A}; \gamma) \not\subset H_\delta^+(\rho_0 - \epsilon) \ .$$

Take one of the vertices $\boldsymbol{v} = (\alpha_k, k)$ of $\mathcal{N}(\omega; \mathcal{A}; \gamma)$ given by Lemma 11. Note that $\boldsymbol{v}$ is not a vertex of $\operatorname{Dom}\mathcal{N}$, hence we can apply the reduction of vertices. We obtain that

$$\mathcal{N}(\omega; \mathcal{A}', \gamma) \subset \mathcal{N}' = \mathcal{N} \setminus \text{interior of } \mathcal{T} \ ,$$

where $\mathcal{T}$ is the triangle having vertices $\boldsymbol{v}$, $(\alpha_{k-1}, k-1)$ and $(\alpha_{k+1}, k+1)$. Moreover

$$\operatorname{area}(\mathcal{T}) = \frac{\delta_r(\boldsymbol{v}) - \delta_l(\boldsymbol{v})}{2} > K_\epsilon/2 \ .$$

We deduce that the area of

$$\mathcal{N}(\omega; \mathcal{A}; \gamma) \cap H_\delta^-(\rho_\delta)$$

decreases strictly at least the amount $K_\epsilon/2$. This cannot be repeated infinitely many times, thus we obtain the condition stated in Lemma 10.

6.6 Elimination of recessive vertices

In this section we complete the proof of Theorem 6. In view of Remark 19 it is enough with determine a ℓ-nested transformation $\mathcal{A} \to \mathcal{B}$ such that $\mathcal{N}(\omega; \mathcal{B}, \gamma) = \operatorname{Dom}\mathcal{N}(\omega; \mathcal{B}, \gamma) = \operatorname{Dom}\mathcal{N}$ and then use Lemma 8.

We assume that the properties listed in Proposition 9 are satisfied. Note that this reductions and Lemma 10 guarantee that for levels ω_k which are not τ_k-final we have

$$\alpha_k \leq \beta_k = \delta_k \leq \tau_k \leq \phi_k \ , \tag{6.4}$$

where we recall that $\beta_k = \nu_{\mathcal{A}}(\omega_k)$, $\delta_k = \nu_{\mathcal{A}}(\eta_k)$ and $\phi_k = \nu_{\mathcal{A}}(f_k)$, and α_k and τ_k are the minimum values such that (α_k, k) and (τ_k, k) belong to $\mathcal{N}$ and Dom$\mathcal{N}$ respectively.

Let us state the induction property we are going to use:

"$\mathcal{P}_h(\omega; \mathcal{A}; \gamma)$: for all $0 \leq k \leq h$ the k-level ω_k is τ_k-final."

Note that ω is γ-prepared if and only if $\mathcal{P}_h(\omega; \mathcal{A}; \gamma)$ is true for all $h \leq \gamma/\nu(z)$.

The starting property $\mathcal{P}_0(\omega; \mathcal{A}; \gamma)$ is true, since $\tau_0 \leq \gamma$ hence ω_0 is τ_0-final. Moreover, the stability properties under ℓ-nested transformations give that

$$\mathcal{P}_h(\omega; \mathcal{A}; \gamma) \Rightarrow \mathcal{P}_h(\omega; \mathcal{A}'; \gamma) \ ,$$

for any ℓ-nested transformation $\mathcal{A} \to \mathcal{A}'$.

Thus, in order to complete the proof of Theorem 6 we have to show the following statement:

> "For a given integer h with $1 \leq h \leq \gamma/\nu(z)$, if $\mathcal{P}_{h-1}(\omega; \mathcal{A}; \gamma)$ is true, there is a ℓ-nested transformation $\mathcal{A} \to \mathcal{A}'$ such that $\mathcal{P}_h(\omega; \mathcal{A}'; \gamma)$ is true."

We suppose that $\mathcal{P}_h(\omega; \mathcal{A}; \gamma)$ is not true. Note that the level ω_h is not $(\gamma - h\nu(z))$-final dominant, otherwise it would be τ_h-final.

Let $L_\delta(\rho)$ be the line passing through the point (τ_h, h) and containing a side of Dom$\mathcal{N}$. We have two possibilities:

a) for every $k < h$ the level ω_k is $(\rho - k\delta)$-final recessive;

b) there is at least on index $k < h$ such that ω_k is $(\rho - k\delta)$-final dominant.

Note that in the first case we must have $\rho = \gamma$. Let us refer to the case a) as the "totally recessive case" and to the case b) as "dominant base point case".

6.6.1 Totally recessive case

In this case we will use Lemma 10 in order to bring $\mathcal{N}$ close enough to $L_\delta(\rho)$ such that the property of reduction of vertices applied to (β_h, h) gives us the desired result.

For every $k < h$ we have that $\tau_k = \rho - k\delta$, thus the real number ϵ given by

$$\epsilon := \min\{\beta_k - \tau_k \mid k = 0, 1, \ldots, h-1\}$$

is strictly positive. Let $\mathcal{A} \to \mathcal{A}^*$ be a ℓ-nested transformation given by Lemma 10 with respect to $L_\delta(\rho)$ and ϵ. Now, the property of reduction of vertices applied to the vertex (β_h, h) of $\mathcal{N}(\omega; \mathcal{A}^*, \gamma)$ gives a ℓ-nested transformation $\mathcal{A}^* \to \mathcal{A}'$ such that $\mathcal{P}_h(\omega; \mathcal{A}'; \gamma)$ is true.

6.6.2 Dominant base point case

Let b be the lowest index such that ω_b is $(\rho - b\delta)$-final dominant. Note that if $\rho < \gamma$ the point (τ_b, b) is a vertex of $\mathrm{Dom}\mathcal{N}$ but in the case $\rho = \gamma$ it is not necessarily a vertex. Taking into account these possibilities, in the case that $b \geq 1$ we define a value $\epsilon_1 > 0$ as

$$\epsilon_1 := \begin{cases} \tau_{b-1} - (\tau_b + \delta) & \text{if} \quad \rho < \gamma \ ; \\ \min\{\beta_k - (\tau_k + (h-k)\delta) \mid k = 0, 1, \ldots, h-1\} & \text{if} \quad \rho = \gamma \ . \end{cases}$$

In the first case ϵ_1 is the distance between $L_\delta(\rho)$ and $\mathrm{Dom}\mathcal{N}$ over the horizontal line at height $b-1$. In the second case, $\beta_k - (\tau_k + (h-k)\delta)$ is the distance between the line $L_\delta(\rho)$ and the point (β_k, k) of $\mathrm{DomCl}(\omega; \mathcal{A})$ over the horizontal line at height k, so ϵ_1 is the minimum among such distances.

Since ω_b is τ_b-final dominant we have

$$\omega_b = \boldsymbol{x}^{\boldsymbol{p}_b}\tilde{\omega}_b + \bar{\omega}_b \ ,$$

where $\nu_\mathcal{A}(\bar{\omega}_b) > \nu(\boldsymbol{x}^{\boldsymbol{p}_b}) = \tau_b$ and $\tilde{\omega}_b$ is log-elementary. If we write $\omega_b = \eta_b + f_b\frac{dz}{z}$ we have

$$\eta_b = \boldsymbol{x}^{\boldsymbol{p}_b}\tilde{\eta}_b + \bar{\eta}_b \quad \text{and} \quad f_b = \tilde{f}_b\boldsymbol{x}^{\boldsymbol{p}_b} + \bar{f}_b \ ,$$

where $\tilde{\eta}_b \in N_\mathcal{A}^\ell$ is log-elementary or $\tilde{\eta}_b \equiv 0$ and $\nu_\mathcal{A}(\bar{\eta}_b) \geq \epsilon$, $\tilde{f}_b \in R_\mathcal{A}^\ell$ is a unit or $\tilde{f}_b \equiv 0$ $\nu_\mathcal{A}(\bar{f}_b) \geq \epsilon$, and $(\tilde{\eta}_b, \tilde{f}_b) \neq (0,0)$. Moreover, we can assume that $\tilde{f}_b$ is a constant, just by taking the non-constant terms and considering them as terms of $\bar{f}_b$, and using Lemma 8 if necessary. So from now on we assume that

$$\omega_b = \boldsymbol{x}^{\boldsymbol{p}_b}\left(\tilde{\eta}_b + \mu\frac{dz}{z}\right) + \bar{\omega}_b \ .$$

Let ϵ_2 be the positive value defined by

$$\epsilon_2 := \nu_\mathcal{A}(\bar{\omega}_b) - \tau_b \ .$$

Consider the value

$$\epsilon := \begin{cases} \min\{\epsilon_1, \epsilon_2\} & \text{if} \quad b \geq 1 \ ; \\ \epsilon_2 & \text{if} \quad b = 0 \ . \end{cases}$$

After performing a ℓ-nested transformation given by Lemma 10 with respect to $L_\delta(\rho)$ and ϵ if necessary, we can assume that:

a) $\beta_k > \tau_h - (k-h)\delta - \epsilon$, for any $h \leq k \leq \gamma/\nu(z)$.

b) $\beta_k \geq \tau_k = \tau_h - (k-h)\delta$, for any $b \leq k \leq h-1$.

c) $\beta_k > \tau_h - (k-h)\delta + \epsilon$, for any $0 \leq k \leq b-1$.

Recall that $\nu_\mathcal{A}(\Delta_{h+b}) \geq 2\gamma$, where this 2-form is given by

$$\Delta_{h+b} = \sum_{i+j=h+b} \left(j\eta_j \wedge \eta_i + f_i d\eta_j + \eta_i \wedge df_j\right) \ .$$

We can write

$$\begin{aligned} &(h-b)\eta_h \wedge \eta_b + f_h d\eta_b + \eta_b \wedge df_h + f_b d\eta_h + \eta_h \wedge df_b = \\ &\qquad = \Delta_{h+b} - \sum_{\substack{i+j=h+b \\ i,j \neq h,b}} \left(j\eta_j \wedge \eta_i + f_i d\eta_j + \eta_i \wedge df_j\right) \end{aligned}$$

In view of properties a), b) and c), and taking into account that $\tau_h + \tau_b < 2\gamma$, we deduce that

$$\nu_{\mathcal{A}}\big((h-b)\eta_h \wedge \eta_b + f_h\, d\eta_b + \eta_b \wedge df_h + f_b\, d\eta_h + \eta_h \wedge df_b\big) \geq \tau_h + \tau_b \ . \tag{6.5}$$

By (6.4) we know that $\nu_{\mathcal{A}}(f_h) \geq \tau_h$. On the other hand, since $\nu_{\mathcal{A}}(\eta_b) \geq \tau_b$ we have that $\nu_{\mathcal{A}}(d\eta_b) \geq \tau_b$. We deduce that

$$\nu_{\mathcal{A}}\left(f_h\, d\eta_b + \eta_b \wedge df_h\right) \geq \tau_h + \tau_b \ . \tag{6.6}$$

From (6.5) and (6.6) we derive that

$$\nu_{\mathcal{A}}\big((h-b)\eta_h \wedge \eta_b + f_b\, d\eta_h + \eta_h \wedge df_b\big) \geq \tau_h + \tau_b \ . \tag{6.7}$$

By property a) we have

$$\nu_{\mathcal{A}}(\eta_h \wedge \bar{\eta}_b) \geq \tau_b + \tau_h \quad \text{and} \quad \nu_{\mathcal{A}}(\bar{f}_b\, d\eta_h) \geq \tau_b + \tau_h \ ,$$

so from Equation (6.7) we deduce

$$\nu_{\mathcal{A}}\left((h-b)\,\eta_h \wedge \boldsymbol{x}^{\boldsymbol{p}_b}\tilde{\eta}_b + \mu\, \boldsymbol{x}^{\boldsymbol{p}_b}\, d\eta_h\right) \geq \tau_h + \tau_b \ .$$

We can rewrite the above expression as

$$\nu_{\mathcal{A}}\left(\boldsymbol{x}^{\boldsymbol{p}_b}\left(\eta_h \wedge \left[(h-b)\tilde{\eta}_b + \mu\frac{d\boldsymbol{x}^{\boldsymbol{p}_b}}{\boldsymbol{x}^{\boldsymbol{p}_b}}\right] + \mu\, d\eta_h\right)\right) \geq \tau_h + \tau_b \ .$$

Dividing by $\boldsymbol{x}^{\boldsymbol{p}_b}$ we obtain

$$\nu_{\mathcal{A}}\left(\eta_h \wedge \left[(h-b)\tilde{\eta}_b + \mu\frac{d\boldsymbol{x}^{\boldsymbol{p}_b}}{\boldsymbol{x}^{\boldsymbol{p}_b}}\right] + \mu\, d\eta_h\right) \geq \tau_h \ . \tag{6.8}$$

Let us denote by $\sigma \in N^{\ell}_{\mathcal{A}}$ the term in the brackets

$$\sigma := (h-b)\tilde{\eta}_b + \mu\frac{d\boldsymbol{x}^{\boldsymbol{p}_b}}{\boldsymbol{x}^{\boldsymbol{p}_b}} \ .$$

Now we study separately two cases depending on whether or not σ is log-elementary.

a) σ is log-elementary. We study first two particular cases first and then we treat the general situation. The first particular case is $\mu = 0$ and the second one is $\tilde{\eta}_b = 0$.

a1) Case $\mu = 0$. In this situation we have that $\sigma = (h-b)\tilde{\eta}_b$ so Equation (6.8) gives

$$\nu_{\mathcal{A}}\left(\eta_h \wedge \tilde{\eta}_b\right) \geq \tau_h \ . \tag{6.9}$$

We need the following lemma:

Lemma 12 (Truncated proportionality)**.** *Let $\tilde{\eta} \in N^{\ell}_{\mathcal{A}}$ be a log-elementary* 1-*form. Given $\theta \in N^{\ell}_{\mathcal{A}}$ such that $\nu_{\mathcal{A}}(\theta \wedge \tilde{\eta}) \geq \lambda$, there is $g \in R^{\ell}_{\mathcal{A}}$ and $\bar{\theta} \in N^{\ell}_{\mathcal{A}}$ with $\nu_{\mathcal{A}}(\bar{\theta}) \geq \lambda$ such that*

$$\theta = g\,\tilde{\eta} + \bar{\theta} \ .$$

Proof. Let us write

$$\tilde{\eta} = \sum a_i \frac{dx_i}{x_i} + \sum b_j dy_j \quad \text{and} \quad \theta = \sum a_i' \frac{dx_i}{x_i} + \sum b_j' dy_j \ ,$$

where we suppose without lost of generality that a_1 is a unit. The coefficients of the 2-form $\theta \wedge \tilde{\eta}$ are given by the minors of the matrix

$$\begin{pmatrix} a_1 & \dots & a_r & b_1 & \dots & b_s \\ a_1' & \dots & a_r' & b_1' & \dots & b_s' \end{pmatrix} .$$

Since $\nu_{\mathcal{A}}(\tilde{\eta} \wedge \theta) \geq \lambda$ we have that

$$\nu_{\mathcal{A}}(a_1 a_i' - a_i a_1') \geq \lambda \ , \quad 2 \leq i \leq r \ ,$$

and

$$\nu_{\mathcal{A}}(a_1 b_j' - b_j a_1') \geq \lambda \ , \quad 1 \leq j \leq n - r \ .$$

Thus we have

$$a_i' = \frac{a_1'}{a_1} a_i + \bar{a}_i \ , \quad \nu_{\mathcal{A}}(\bar{a}_i) \geq \lambda \ , \quad 2 \leq i \leq r \ ,$$

and

$$b_j' = \frac{a_1'}{a_1} b_j + \bar{b}_j \ , \quad \nu_{\mathcal{A}}(\bar{b}_j) \geq \lambda \ , \quad 1 \leq j \leq n - r \ .$$

Therefore we can write

$$\theta = g\,\tilde{\eta} + \bar{\theta} \ ,$$

where

$$g = \frac{a_1'}{a_1} \quad \text{and} \quad \bar{\theta} = \sum \bar{a}_i \frac{dx_i}{x_i} + \sum \bar{b}_j dy_j \ ,$$

and by construction we have $\nu_{\mathcal{A}}(\bar{\theta}) \geq \lambda$. □

Remark 20. We have detailed a direct proof of Lemma 12, but it can be obtained as a consequence of the de Rham-Saito Lemma [20].

From Equation (6.9) and Lemma 12 we conclude that there are $g \in R_{\mathcal{A}}^{\ell}$ and $\bar{\eta}_h \in N_{\mathcal{A}}^{\ell}$ such that

$$\eta_h = g\sigma + \bar{\eta}_h \ ,$$

where $\nu_{\mathcal{A}}(\bar{\eta}_h) \geq \tau_h$. This expression is stable under further ℓ-nested transformations, thus we can assume that g is τ_h-final. If $\nu_{\mathcal{A}}(g) < \tau_h$ the level ω_h would be τ_h-final dominant with value $\nu_{\mathcal{A}}(\omega_h) = \nu_{\mathcal{A}}(g) < \tau_h$, in contradiction with the maximality of dominant levels assumption. So we have $\nu_{\mathcal{A}}(g) \geq \tau_h$, hence $\nu_{\mathcal{A}}(\omega_h) \geq \tau_h$. Since ω_h cannot be dominant, applying Lemma 8 if necessary we obtain $\nu_{\mathcal{A}}(\omega_h) > \tau_h$, that is, ω_h is τ_h-final recessive.

a2) Case $\tilde{\eta}_b = 0$. In this situation we have $\mu \neq 0$. In this situation we have that

$$\sigma = \mu \frac{d\boldsymbol{x}^{\boldsymbol{p}_b}}{\boldsymbol{x}^{\boldsymbol{p}_b}} \ ,$$

hence Equation (6.8) gives

$$\nu_{\mathcal{A}}\left(\eta_h \wedge \frac{d\boldsymbol{x}^{\boldsymbol{p}_b}}{\boldsymbol{x}^{\boldsymbol{p}_b}} + d\eta_h\right) \geq \tau_h \ . \tag{6.10}$$

Let us write η_h as a series in the independent variables

$$\eta_h = \sum_I \boldsymbol{x}^I \eta_{h,I} \ ,$$

where the coefficients of the 1-forms $\eta_{h,I} \in N^\ell_{\mathcal{A}}$ are series in the variables $\boldsymbol{y}$. Let us denote $\eta_h = \check{\eta}_h + \bar{\eta}_h$ where

$$\check{\eta}_h = \sum_{\nu(\boldsymbol{x}^I)<\tau_h} \boldsymbol{x}^I \eta_{h,I} \ . \tag{6.11}$$

Since $\nu_{\mathcal{A}}(\bar{\eta}_h) \geq \tau_h$, from Equation (6.10) we have that

$$\nu_{\mathcal{A}}\left(\check{\eta}_h \wedge \frac{d\boldsymbol{x}^{\boldsymbol{p}_b}}{\boldsymbol{x}^{\boldsymbol{p}_b}} + d\check{\eta}_h\right) \geq \tau_h \ . \tag{6.12}$$

Note that this expression is homogeneous with respect to $\boldsymbol{x}$, it means,

$$\check{\eta}_h \wedge \frac{d\boldsymbol{x}^{\boldsymbol{p}_b}}{\boldsymbol{x}^{\boldsymbol{p}_b}} + d\check{\eta}_h = \sum_{\nu(\boldsymbol{x}^I)<\tau_h} \boldsymbol{x}^I \left(\eta_{h,I} \wedge \frac{d\boldsymbol{x}^{\boldsymbol{p}_b}}{\boldsymbol{x}^{\boldsymbol{p}_b}} + \frac{d\boldsymbol{x}^{\boldsymbol{I}}}{\boldsymbol{x}^{\boldsymbol{I}}} \wedge \eta_{h,I} + d\eta_{h,I}\right) \ .$$

Due to this homogeneity we have that Equation (6.12) is equivalent to

$$\check{\eta}_h \wedge \frac{d\boldsymbol{x}^{\boldsymbol{p}_b}}{\boldsymbol{x}^{\boldsymbol{p}_b}} + d\check{\eta}_h = 0 \ . \tag{6.13}$$

Multiplying by $\check{\eta}_h$ the above expression we deduce that $\check{\eta}_h$ is integrable. By the induction statement $T_3(\ell)$ there is a ℓ-nested transformation $\mathcal{A} \to \mathcal{A}'$ such that $\check{\eta}_h$ is τ_h-final. If $\nu_{\mathcal{A}'}(\check{\eta}_h) < \tau_h$ the level ω_h would be τ_h-final dominant with value $\nu_{\mathcal{A}'}(\omega_h) = \nu_{\mathcal{A}'}(\check{\eta}_h) < \tau_h$, in contradiction with the maximality of dominant levels assumption. So we have $\nu_{\mathcal{A}'}(\check{\eta}_h) \geq \tau_h$, hence $\nu_{\mathcal{A}}(\omega_h) \geq \tau_h$. Since ω_h cannot be dominant, applying Lemma 8 if necessary we obtain $\nu_{\mathcal{A}'}(\omega_h) > \tau_h$, that is, ω_h is τ_h-final recessive.

a3) General case. Let us write

$$\sigma = (h-b)\tilde{\eta}_b + \mu \frac{d\boldsymbol{x}^{\boldsymbol{p}_b}}{\boldsymbol{x}^{\boldsymbol{p}_b}} = \sum_{i=1}^r \lambda_i \frac{dx_i}{x_i} + \sigma^* \ ,$$

where $\boldsymbol{\lambda} \in k^r \setminus \{\boldsymbol{0}\}$ and $\sigma^* \in N^\ell_{\mathcal{A}}$ is not log-elementary. Suppose that we perform a ℓ-nested transformation $\mathcal{A} \to \mathcal{A}'$. We obtain new coordinates $(\boldsymbol{x}', \boldsymbol{y}')$ such that for $i = 1, 2, \ldots, r$ we have

$$x_i = \boldsymbol{x}'^{\boldsymbol{\alpha}_i} U_i \ ,$$

where $\boldsymbol{\alpha}_i \in \mathbb{Z}^r_{\geq 0} \setminus \{\boldsymbol{0}\}$ and $U_i \in R^\ell_{\mathcal{A}'}$ is a unit. We have that

$$\sum_{i=1}^r \lambda_i \frac{dx_i}{x_i} = \sum_{i=1}^r \lambda_i \frac{d(\boldsymbol{x}'^{\boldsymbol{\alpha}_i} U_i)}{\boldsymbol{x}'^{\boldsymbol{\alpha}_i} U_i} = \sum_{i=1}^r \lambda_i \frac{d\boldsymbol{x}'^{\boldsymbol{\alpha}_i}}{\boldsymbol{x}'^{\boldsymbol{\alpha}_i}} + \sum_{i=1}^r \lambda_i U_i^{-1} dU_i \ .$$

We can write

$$\sum_{i=1}^r \lambda_i \frac{d\boldsymbol{x}'^{\boldsymbol{\alpha}_i}}{\boldsymbol{x}'^{\boldsymbol{\alpha}_i}} = \sum_{i=1}^r \lambda'_i \frac{dx'_i}{x'_i}$$

where $\boldsymbol{\lambda}' \in k^r \setminus \{\mathbf{0}\}$. On the other hand, we have that

$$d\left(\sum_{i=1}^{r} \lambda_i U_i^{-1} dU_i\right) = 0 \ ,$$

so, by Poincare's Lemma we know that

$$\sum_{i=1}^{r} \lambda_i U_i^{-1} dU_i = dG$$

for certain formal function $G \in R^{\ell}_{\mathcal{A}'}$. We have just proved that after performing a ℓ-nested transformation $\mathcal{A} \to \mathcal{A}'$ the 1-form σ can be written as

$$\sigma = \sum_{i=1}^{r} \lambda_i' \frac{dx_i'}{x_i'} + dG + \sigma^* \ .$$

In view of these considerations and using if necessary Lemmas 8 and 10 we can assume that in $\mathcal{A}$ we have

$$\sigma = \sum_{i=1}^{r} \lambda_i \frac{dx_i}{x_i} + dG + \sigma^* \ ,$$

where $\nu_{\mathcal{A}}(\sigma^*) > \tau_h - \nu_{\mathcal{A}}(\eta_h) > 0$. With this assumption we have that Equation (6.8) gives

$$\nu_{\mathcal{A}}\left(\eta_h \wedge \left[\sum_{i=1}^{r} \lambda_i \frac{dx_i}{x_i} + dG\right] + \mu\, d\eta_h\right) \geq \tau_h \ . \tag{6.14}$$

Moreover, we can assume that $\lambda_{i_0} = 1$ for certain index $1 \leq i_0 \leq r$. Let us consider the elements $x_1^*, x_2^*, \ldots, x_r^* \in R^{\ell}_{\mathcal{A}}$ defined by $x_i^* := x_i$ if $i \neq i_0$ and $x_{i_0}^* = x_{i_0} \exp(G)$. Note that we have

$$\sum_{i=1}^{r} \lambda_i \frac{dx_i}{x_i} + dG = \sum_{i=1}^{r} \lambda_i \frac{dx_i^*}{x_i^*} \ .$$

We have that $x_1^*, x_2^*, \ldots, x_r^*, y_1, y_2, \ldots, y_\ell$ are a regular system of parameters of $R^{\ell}_{\mathcal{A}}$. Let us consider a new "explicit value" $\nu^*_{\mathcal{A}}$, defined exactly as $\nu_{\mathcal{A}}$ but considering the power series expansions with respect to the parameters $\boldsymbol{x}^*$ instead of $\boldsymbol{x}$. For every exponent $\boldsymbol{q} \in \mathbb{Z}^r_{\geq 0}$ it follows that

$$\nu_{\mathcal{A}}(\boldsymbol{x}^{*\boldsymbol{q}}) = \nu(\boldsymbol{x}^{\boldsymbol{q}}) \ ,$$

so we have that $\nu^*_{\mathcal{A}} \equiv \nu_{\mathcal{A}}$. It means, for any object ψ (formal function or p-form) we have that $\nu^*_{\mathcal{A}}(\psi) = \nu_{\mathcal{A}}(\psi)$.

Let us write η_h as a power series of the elements $x_1^*, x_2^*, \ldots, x_r^*$

$$\eta_h = \sum_{I} \boldsymbol{x}^{*I} \eta_{h,I}$$

where the coefficients of the 1-forms $\eta_{h,I} \in N^{\ell}_{\mathcal{A}}$ are series in the variables $\boldsymbol{y}$. Let us denote $\eta_h = \check{\eta}_h + \bar{\eta}_h$ where

$$\check{\eta}_h = \sum_{\nu^*_{\mathcal{A}}(\boldsymbol{x}^{*I}) < \tau_h} \boldsymbol{x}^{*I} \eta_{h,I} \ .$$

We have that $\check{\eta}_h$ and $\bar{\eta}_h$ belong to $N^{\ell}_{\mathcal{A}}$ and $\nu^*_{\mathcal{A}}(\bar{\eta}_h) \geq \tau_h$. From Equation (6.14) we obtain

$$\nu^*_{\mathcal{A}}\left(\check{\eta}_h \wedge \sum_{i=1}^{r} \lambda_i \frac{dx_i^*}{x_i^*} + \mu\, d\check{\eta}_h\right) \geq \tau_h \ .$$

This expression is homogeneous with respect to $\boldsymbol{x}^*$ so it is equivalent to

$$\check{\eta}_h \wedge \sum_{i=1}^{r} \lambda_i \frac{dx_i^*}{x_i^*} + \mu\, d\check{\eta}_h = 0 \ .$$

Multiplying this last expression by $\check{\eta}_h$ we obtain that $\check{\eta}_h$ is an integrable 1-form. Recall that both $\check{\eta}_h$ and $\bar{\eta}_h$ belong to $N^{\ell}_{\mathcal{A}}$ and that $\nu_{\mathcal{A}}(\bar{\eta}_h) \geq \tau_h$. We are in the same situation that in the particular case $\tilde{\eta}_b \equiv 0$ detailed previously, so we can end as we did in that case.

Remark 21. Note that we have not performed non-algebraic operations. We have used the formal variables $\boldsymbol{x}^*$ only for divide in two parts the 1-form η_h.

b) σ is not log-elementary. First of all, note that $\mu = 0$ implies that σ is log-elementary, so in this case we have $\mu \neq 0$.

In this case, using Lemma 8 if necessary, we can assume that $\nu_{\mathcal{A}}(\sigma) > 0$. In fact, we can suppose that $\nu_{\mathcal{A}}(\sigma) > \tau_h - \nu_{\mathcal{A}}(\eta_h)$, otherwise we use Lemma 10. With these assumptions Equation (6.8) gives

$$\nu_{\mathcal{A}}(d\eta_h) \geq \tau_h \ .$$

Writing $\eta_h = \check{\eta}_h + \bar{\eta}_h$ as we did in Equation (6.11), where we recall that $\nu_{\mathcal{A}}(\bar{\eta}_h) \geq \tau_h$, we obtain that

$$\nu_{\mathcal{A}}(d\check{\eta}_h) \geq \tau_h \ ,$$

and again due to the homogeneity with respect to the variables $\boldsymbol{x}$ we have that

$$d\check{\eta}_h = 0 \ .$$

We have that $\check{\eta}_h$ is integrable (indeed, following Poincare's Lemma it is the differential of a function), and we conclude this case as the previous one.

Chapter 7

Getting γ-final forms

In this chapter we complete the proof of the induction step

$$T_3(\ell) \Longrightarrow T_3(\ell+1)$$

started in Chapter 6, thus we also end the proof of Theorem 3.

Let $\mathcal{A} = \big(\mathcal{O}, (\boldsymbol{x}, \boldsymbol{y})\big)$ be a parameterized regular local model for K, ν. Fix an index ℓ, $0 \leq \ell \leq n-r-1$. Let us recall here the precise statement we want to prove:

> $\boldsymbol{T_3(\ell+1)}$: Given a 1-form $\omega \in N_{\mathcal{A}}^{\ell+1}$ and a value $\gamma \in \Gamma$, if $\nu_{\mathcal{A}}(\omega \wedge d\omega) \geq 2\gamma$ then there exists a $(\ell+1)$-nested transformation $\mathcal{A} \to \mathcal{B}$ such that ω is γ-final in $\mathcal{B}$.

So, during this chapter we fix a value $\gamma \in \Gamma$ and consider a 1-form $\omega \in N_{\mathcal{A}}^{l+1}$ such that $\nu_{\mathcal{A}}(\omega \wedge d\omega) \geq 2\gamma$. Since we are working by induction on ℓ, we assume that the statement $T_3(\ell)$ is true (hence $T_4(\ell)$ and $T_5(\ell)$ are also true).

As in the previous chapter we denote the dependent variables by $\boldsymbol{y} = (y_1, y_2, \ldots, y_\ell)$ and $z = y_{\ell+1}$.

7.1 The critical height of a γ-prepared 1-form

In this section we assume that $\omega \in N_{\mathcal{A}}^{\ell+1}$ is γ-prepared. Recall that due to the γ-prepared assumption in this situation we have that $\mathcal{N}(\omega; \mathcal{A}; \gamma) = \operatorname{Dom} \mathcal{N}(\omega; \mathcal{A}; \gamma)$.

The *critical value* $\delta(\omega; \mathcal{A}; \gamma)$ is defined by

$$\delta(\omega; \mathcal{A}; \gamma) := \min \left\{ \rho \mid L_{\nu(z)}(\rho) \cap \mathcal{N}(\omega; \mathcal{A}; \gamma) \neq \emptyset \right\} .$$

Note that $\delta(\omega; \mathcal{A}; \gamma) \leq \gamma$ since $(0, \gamma) \in \mathcal{N}(\omega; \mathcal{A}; \gamma)$. The critical value satisfies

$$\delta(\omega; \mathcal{A}; \gamma) = \min \left\{ \beta_k(\omega; \mathcal{A}) + k\nu(z) \right\}_{k \geq 0} \cup \{\gamma\}$$

where we recall that $\beta_k(\omega; \mathcal{A}) = \nu_{\mathcal{A}}(\omega_k)$ (see Section 6.1). Note that due to the γ-preparation assumption in the above equality we can put τ_k instead of β_k. If no confusion arises we denote the critical value by δ. We study separately the cases $\delta < \gamma$ and $\delta = \gamma$

In the case $\delta < \gamma$ we say that $\mathcal{N}(\omega;\mathcal{A};\gamma) \cap L_{\nu(z)}(\delta)$ is *the critical segment of* $\mathcal{N}(\omega;\mathcal{A};\gamma)$. The *critical height* $\chi(\omega;\mathcal{A};\gamma)$ *of* $\mathcal{N}(\omega;\mathcal{A};\gamma)$ is the height of the upper endpoint of the critical segment. This integer number is our main control invariant. It satisfies

$$0 \leq \chi(\omega;\mathcal{A};\gamma) \leq \frac{\delta}{\nu(z)} < \frac{\gamma}{\nu(z)} .$$

If no confusion arises we denote the critical height by χ. Note that we have

$$\delta = \tau_\chi + \chi\,\nu(z) = \beta_\chi + \chi\,\nu(z) . \tag{7.1}$$

Denote by $\beta(\omega;\mathcal{A})$ the explicit value $\nu_{\mathcal{A}}(\omega)$ of ω in $\mathcal{A}$. Note that $\beta(\omega;\mathcal{A})$ is the minimum of the values $\beta_k(\omega;\mathcal{A})$. If $\delta(\omega;\mathcal{A};\gamma) < \gamma$, from Equation (7.1) we derive that

$$\delta(\omega;\mathcal{A};\gamma) \geq \beta(\omega;\mathcal{A}) + \chi(\omega;\mathcal{A};\gamma)\nu(z) , \tag{7.2}$$

where we have equality if and only if $\beta(\omega;\mathcal{A})$ is the abscissa of the critical vertex. If no confusion arises we denote the explicit value of ω by β.

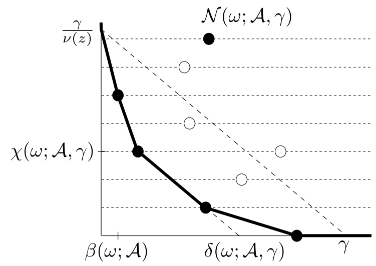

Figure 7.1: The explicit value, the critical value and the critical height

Denote by $\boldsymbol{q}_k \in \mathbb{Z}^r_{\geq 0}$ the exponent such that $\nu(\boldsymbol{x}^{\boldsymbol{q}_k}) = \beta_k$. A level ω_k gives a point $(\beta_k, k) = (\tau_k, k)$ in the critical segment if and only if

$$\beta_k = \tau_k = \delta - k\nu(z) = \tau_\chi + (\chi - k)\nu(z) .$$

If it is the situation the index k must be of the form $k = \chi - td$ for some integer $0 \leq t \leq \chi/d$, where $d = d(\ell+1,\mathcal{A})$. Following Remark 7 this is equivalent to

$$z^k \boldsymbol{x}^{\boldsymbol{q}_k} = z^\chi \boldsymbol{x}^{\boldsymbol{q}_\chi} \phi^{-t} ,$$

where $\phi = z^d/\boldsymbol{x}^{\boldsymbol{p}}$ is the $(\ell+1)$-th contact rational function in $\mathcal{A}$. For any index $k = \chi - td$ such that (β_k, k) is a point of the critical segment, let us define $\sigma_{\mathcal{A},t} \in N^\ell_{\mathcal{A}} \oplus R^\ell_{\mathcal{A}} \frac{dz}{z}$ as the 1-form with constant coefficients given by

$$\sigma_{\mathcal{A},t} := \frac{1}{\boldsymbol{x}^{\boldsymbol{q}_{\chi-td}}} \,\mathrm{in}_{\mathcal{A}}(\omega_{\chi-td}) ,$$

where we recall that $\mathrm{in}_{\mathcal{A}}(\omega_{\chi-td})$ was defined in Equation (6.2). We put $\sigma_{\mathcal{A},t} := 0$ for indices t such that $\omega_{\chi-td}$ does not give a point in the critical segment. We define the $\mathcal{A}$-*critical part of* ω by

$$\mathrm{crit}_{\mathcal{A}}(\omega) := \boldsymbol{x}^{\boldsymbol{q}_\chi} z_\chi \sum_{t=0}^{M} \phi^{-t} \sigma_{\mathcal{A},t} , \tag{7.3}$$

where M denotes the integer part of χ/d.

Remark 22. Note that $\mathrm{crit}_{\mathcal{A}}(\omega)$ is the sum of the $\mathcal{A}$-initial parts of the levels corresponding to the critical segment.

After performing a $(\ell+1)$-Puiseux's package we obtain a parameterized regular local model in which the 1-form ω not need to be γ-prepared. By Theorem 6 there is a γ-preparation, so we will perform it and compare the new critical value and height with the old ones.

7.2 Pre-γ-final 1-forms

As we see in Section 5.3 in the case of functions, if the critical value is γ or if the critical height is 0 we know how to obtain a γ-final situation. The same happens when we deal with 1-forms.

Definition 21. A γ-prepared 1-form $\omega \in N_{\mathcal{A}}^{\ell+1}$ is *pre-γ-final* if

$$\delta(\omega;\mathcal{A};\gamma)=\gamma$$

or

$$\delta(\omega;\mathcal{A};\gamma)<\gamma \quad \text{and} \quad \chi(\omega;\mathcal{A};\gamma)=0\ .$$

Pre-γ-final functions are easily recognizable by its Truncated Newton Polygon as it is represented in Figure 7.2

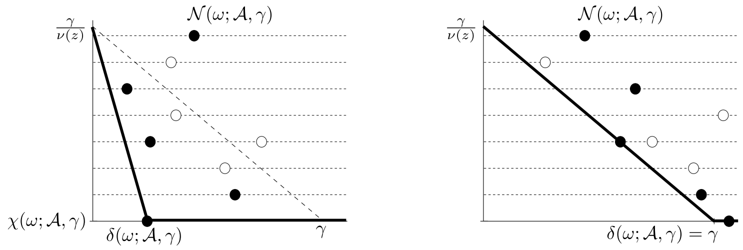

Figure 7.2: The two pre-γ-final situations

Let $\Psi_{\ell+1}$ be the $(\ell+1)$-nested transformation given in Lemma 8.

Proposition 10. *Let $\omega \in N_{\mathcal{A}}^{\ell+1}$ be a pre-γ-final 1-form. Consider the $(\ell+1)$-nested transformation*

$$\mathcal{A} \xrightarrow{\pi} \mathcal{A}' \xrightarrow{\Psi_{\ell+1}} \mathcal{B}$$

where $\pi:\mathcal{A}\to\mathcal{A}'$ is a $(\ell+1)$-Puiseux's package. Then ω is γ-final in $\mathcal{B}$.

Proof. Consider the decomposition in z-levels of ω in $\mathcal{A}$

$$\omega=\sum_{k=0}^{\infty} z^k\omega_k=\sum_{k=0}^{\infty} z^k\left(\eta_k+f_k\frac{dz}{z}\right)\ .$$

First, suppose we are in the first case $\delta(\omega;\mathcal{A};\gamma)=\gamma$. For each index $k\geq 0$ we have

$$\nu_{\mathcal{A}'}(\omega_k)\geq \nu_{\mathcal{A}}(\omega_k)\geq \gamma-k\nu(z)\ .$$

From Equations (2.3) we know that

$$z = \boldsymbol{x}'^{\boldsymbol{\alpha}_0}(z' + \xi)^{\beta_0} \ , \quad \text{with } \nu(\boldsymbol{x}'^{\boldsymbol{\alpha}_0}) = \nu(z) \ ,$$

hence

$$\nu_{\mathcal{A}'}(z^k) = \nu_{\mathcal{A}'}\left(\boldsymbol{x}'^{k\boldsymbol{\alpha}_0}(z' + \xi)^{k\beta_0}\right) = k\nu(z) \ .$$

Therefore, for each $k \geq 0$ we have

$$\nu_{\mathcal{A}'}(z^k \omega_k) = \nu_{\mathcal{A}'}(z^k) + \nu_{\mathcal{A}'}(\omega_k) \geq \gamma \ .$$

It follows that

$$\beta(\omega; \mathcal{A}') = \nu_{\mathcal{A}'}(\omega) \geq \gamma \ .$$

If $\beta(\omega; \mathcal{A}') > \gamma$ then ω is γ-final recessive in $\mathcal{A}'$ so it is also γ-final recessive in $\mathcal{B}$ (Lemma 6). On the other hand, if $\beta(\omega; \mathcal{A}') = \gamma$, by Lemma 8 ω is γ-final in $\mathcal{B}$.

Now suppose that $\chi(\omega; \mathcal{A}; \gamma) = 0$. All the levels ω_k with $k > 0$ do not belong to the critical segment, so we have

$$\nu_{\mathcal{A}'}(\omega_k) \geq \nu_{\mathcal{A}}(\omega_k) > \delta(\omega; \mathcal{A}; \gamma) - k\nu(z) \ , \quad \forall k \geq 1 \ .$$

It follows that

$$\nu_{\mathcal{A}'}(z^k \omega_k) > \delta(\omega; \mathcal{A}; \gamma) \quad \text{for all } k > 0 \ ,$$

hence

$$\nu_{\mathcal{A}'}(\omega - \omega_0) = \nu_{\mathcal{A}'}\left(\sum_{k=1}^{\infty} z^k \omega_k\right) > \delta(\omega; \mathcal{A}; \gamma) \ . \tag{7.4}$$

Thinking in ω_0 as a element of $N_{\mathcal{A}}^{\ell+1}$, it is γ-final dominant with explicit value $\nu_{\mathcal{A}}(\omega_0) = \delta(\omega; \mathcal{A}; \gamma)$. By Lemma 6 we have that ω_0, as a element of $N_{\mathcal{A}'}^{\ell+1}$, is also γ-final dominant with explicit value

$$\nu_{\mathcal{A}'}(\omega_0) = \delta(\omega; \mathcal{A}; \gamma) \ .$$

Taking into account Equation (7.4) we have that ω is γ-final dominant with explicit value

$$\nu_{\mathcal{A}'}(\omega) = \nu_{\mathcal{A}'}(\omega_0 + (\omega - \omega_0)) = \delta(\omega; \mathcal{A}; \gamma) \ .$$

Finally, it follows from Lemma (6) that the same happens in $\mathcal{B}$. $\square$

7.3 Stability of the Critical Height

In view of Proposition 10, in order to complete the proof of $T_3(\ell+1)$ it is enough with determine a $(\ell+1)$-nested transformation such that ω becomes pre-γ-final. In this section we show that the critical height cannot increase by means of $(\ell+1)$-nested transformations.

Proposition 11. *Let $\omega \in N_{\mathcal{A}}^{\ell+1}$ a γ-prepared 1-form. Consider the $(\ell+1)$-nested transformation*

$$\mathcal{A} \xrightarrow{T} \tilde{\mathcal{B}} \xrightarrow{\tau} \mathcal{B}$$

where $T : \mathcal{A} \to \tilde{\mathcal{B}}$ is an ordered change of the variable z and $\tau : \tilde{\mathcal{B}} \to \mathcal{B}$ is a γ-preparation. Then

$$\beta(\mathcal{B};\omega) = \beta(\mathcal{A};\omega) \quad \textit{and} \quad \delta(\omega;\mathcal{B},\gamma) \geq \delta(\omega;\mathcal{A};\gamma) \ .$$

In addition, if $\delta(\omega;\mathcal{B},\gamma) < \gamma$ we have that

$$\chi(\omega;\mathcal{B},\gamma) \leq \chi(\omega;\mathcal{A};\gamma) \ .$$

Proof. Consider the decomposition in z-levels of ω in $\mathcal{A}$

$$\omega = \sum_{k=0}^{\infty} z^k \omega_k = \sum_{k=0}^{\infty} z^k \left(\eta_k + f_k \frac{dz}{z} \right) \ .$$

The ordered change of variables $T : \mathcal{A} \to \tilde{\mathcal{B}}$ is given by $\tilde{z} := z - \psi$ where ψ is a polynomial $\psi \in k[\boldsymbol{x}, y_1, y_2, \ldots, y_\ell]$ such that $\nu_{\mathcal{A}}(\psi) \geq \nu(z)$. Note that $\nu_{\mathcal{A}} \equiv \nu_{\tilde{\mathcal{B}}}$. In $\tilde{\mathcal{B}}$ the decomposition in $\tilde{z}$-levels is given by

$$\omega = \sum_{k=0}^{\infty} z^k \tilde{\omega}_k = \sum_{k=0}^{\infty} z^k \left(\tilde{\eta}_k + \tilde{f}_k \frac{dz}{z} \right) \ ,$$

where

$$\tilde{\eta}_k = \eta_k + f_{k+1} d\psi + \sum_{j=1}^{\infty} \binom{k+j}{j} \psi^j \left(\eta_{k+j} + f_{k+1+j}\, d\psi \right)$$

and

$$\tilde{f}_k = f_k + \sum_{j=1}^{\infty} \binom{k-1+j}{j} \psi^j f_{k+j} \ .$$

First, suppose we have $\delta(\omega;\mathcal{A};\gamma) = \gamma$. This is equivalent to say that for every $k \geq 0$ we have

$$\nu_{\mathcal{A}}(\omega_k) \geq \gamma - k\nu(z) \ ,$$

hence

$$\nu_{\mathcal{A}}(\eta_k) \geq \gamma - k\nu(z) \quad \text{and} \quad \nu_{\mathcal{A}}(f_k) \geq \gamma - k\nu(z) \ . \tag{7.5}$$

Recall that $\nu_{\mathcal{A}}(\psi) \geq \nu(z)$ implies $\nu_{\mathcal{A}}(d\psi) \geq \nu(z)$, thus in view of (7.5) we have

$$\nu_{\mathcal{A}}(\tilde{\eta}_k) \geq \gamma - k\nu(z) \quad \text{and} \quad \nu_{\mathcal{A}}(\tilde{f}_k) \geq \gamma - k\nu(z) \ ,$$

so

$$\nu_{\mathcal{B}}(\tilde{\omega}_k) = \nu_{\mathcal{A}}(\tilde{\omega}_k) \geq \gamma - k\nu(z) \geq \gamma - k\nu(\tilde{z}) \ ,$$

hence $\delta(\omega;\mathcal{B},\gamma) = \gamma$.

Now, suppose $\delta(\omega;\mathcal{A};\gamma) < \gamma$. For short, denote by χ the critical height $\chi(\omega;\mathcal{A};\gamma)$. Since ω is γ-prepared, for all index $t \geq 1$ we have

$$\nu_{\mathcal{A}}(\omega_{\chi+t}) > \nu_{\mathcal{A}}(\omega_\chi) - t\nu(z) \ ,$$

so

$$\nu_{\mathcal{A}}(\eta_{\chi+t}) > \nu_{\mathcal{A}}(\omega_\chi) - t\nu(z) \quad \text{and} \quad \nu_{\mathcal{A}}(f_{\chi+t}) > \nu_{\mathcal{A}}(\omega_\chi) - t\nu(z) \ . \tag{7.6}$$

From (7.6) we have

$$\nu_{\tilde{\mathcal{B}}}\left(\psi^j\left(\eta_{\chi+t+j}+f_{\chi+t+1+j}\,d\psi\right)\right)>\nu_{\tilde{\mathcal{B}}}(\omega_\chi)-t\nu(z)$$

and

$$\nu_{\tilde{\mathcal{B}}}\left(\psi^j f_{\chi+t+j}\right)>\nu_{\tilde{\mathcal{B}}}(\omega_\chi)-t\nu(z)$$

for all $t\geq 1$ and all $j\geq 1$. Thus we have

$$\nu_{\tilde{\mathcal{B}}}(\tilde{\eta}_{\chi+t})>\nu_{\tilde{\mathcal{B}}}(\omega_\chi)-t\nu(z)$$

and

$$\nu_{\tilde{\mathcal{B}}}(\tilde{f}_{\chi+t})>\nu_{\tilde{\mathcal{B}}}(\omega_\chi)-t\nu(z)$$

hence

$$\nu_{\tilde{\mathcal{B}}}(\tilde{\omega}_{\chi+t})>\nu_{\tilde{\mathcal{B}}}(\omega_\chi)-t\nu(z)\quad\text{for all } t\geq 1\ . \tag{7.7}$$

In the same way we see that

$$\nu_{\tilde{\mathcal{B}}}(\tilde{\omega}_{\chi-t})\geq\nu_{\tilde{\mathcal{B}}}(\omega_\chi)+t\nu(z)\quad\text{for } 1\leq t\leq\chi\ , \tag{7.8}$$

and that $\tilde{\omega}_\chi$ is dominant with explicit value

$$\nu_{\tilde{\mathcal{B}}}(\tilde{\omega}_\chi)=\nu_{\mathcal{A}}(\omega_\chi)\ , \tag{7.9}$$

After performing the γ-preparation $\tilde{\mathcal{B}}\to\mathcal{B}$ we still have the properties given in (7.7), (7.8) and (7.9) replacing $\nu_{\tilde{\mathcal{B}}}$ by $\nu_{\mathcal{B}}$. Let $z'=\tilde{z}$ be the $(\ell+1)$-th dependent variable in $\mathcal{B}$. Since $\nu(z')\geq\nu(z)$ and taking into account (7.7) and (7.9) we have

$$\delta(\omega;\mathcal{B},\gamma)\geq\nu_{\mathcal{B}}(\tilde{\omega}_\chi)+\chi\nu(z')\geq\nu_{\mathcal{A}}(\omega_\chi)+\chi\nu(z)=\delta(\omega;\mathcal{A};\gamma)$$

as desired. In addition, if $\delta(\omega;\mathcal{B},\gamma)<\gamma$, from (7.8) and $\nu(z')\geq\nu(z)$ we have

$$\chi(\omega;\mathcal{B},\gamma)\leq\chi(\omega;\mathcal{A};\gamma)\ .$$

□

Proposition 12. *Let $\omega\in N_{\mathcal{A}}^{\ell+1}$ be a γ-prepared 1-form. Suppose that $\delta(\omega;\mathcal{A};\gamma)<\gamma$. Consider the $(\ell+1)$-nested transformation*

$$\mathcal{A}\xrightarrow{\pi}\tilde{\mathcal{B}}\xrightarrow{\tau}\mathcal{B}$$

where $\pi:\mathcal{A}\to\tilde{\mathcal{B}}$ is a $(\ell+1)$-Puiseux's package and $\tau:\tilde{\mathcal{B}}\to\mathcal{B}$ is a γ-preparation. Then

$$\beta(\mathcal{B};\omega)=\delta(\omega;\mathcal{A};\gamma)\ .$$

In addition, if $\delta(\omega;\mathcal{B},\gamma)<\gamma$ we have that

$$\chi(\omega;\mathcal{B},\gamma)\leq\chi(\omega;\mathcal{A};\gamma)\ .$$

Proof. Denote by M the integer part of $\chi(\omega;\mathcal{A};\gamma)/d(\ell+1,\mathcal{A})$. For $t=0,1,\ldots,M$ let us write

$$\sigma_{\mathcal{A},t}=\sum_{i=1}^{r}\lambda_{t,i}\frac{dx_i}{x_i}+\mu_t\frac{dz}{z}\ ,$$

where we recall from Equation (7.3) that

$$\mathrm{crit}_{\mathcal{A}}(\omega) = \boldsymbol{x}^{\boldsymbol{q}_\chi} z_\chi \sum_{t=0}^{M} \phi^{-t} \sigma_{\mathcal{A},t} \ .$$

Let $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{z})$ be the coordinates in the parameterized regular local model $\tilde{\mathcal{B}}$ obtained from $\mathcal{A}$ by means of a $(\ell+1)$-Puiseux's package. We have

$$\mathrm{crit}_{\mathcal{A}}(\omega) = \tilde{\boldsymbol{x}}^{\boldsymbol{r}} \phi^{e} \sum_{t=0}^{M} \phi^{-t} \sigma_{\mathcal{A},t} \ , \tag{7.10}$$

where $\nu(\tilde{\boldsymbol{x}}^{\boldsymbol{r}}) = \delta(\omega; \mathcal{A}; \gamma)$. The exponents $\boldsymbol{r} \in \mathbb{Z}^{r}_{\geq 0}$ and $e \in \mathbb{Z}_{>0}$ are determined by the equalities given in (2.3). Note that $\phi = \tilde{z} + \xi$ is a unit in $R^{\ell+1}_{\tilde{\mathcal{B}}}$. We can rewrite (7.10) as

$$\mathrm{crit}_{\mathcal{A}}(\omega) = \tilde{\boldsymbol{x}}^{\boldsymbol{r}} U \sum_{t=0}^{M} (\tilde{z} + \xi)^{M-t} \sigma_t \ , \tag{7.11}$$

where $U = U(\tilde{z}) = \phi^{e-M}$. For each index t denote

$$(\lambda'_{t,1}, \ldots, \lambda'_{t,r}, \mu'_t) = (\lambda_{t,1}, \lambda_{t,2}, \ldots, \lambda_{t,r}, \mu_t) H_\pi \ , \tag{7.12}$$

where H_π is the invertible matrix of non-negative integers corresponding to the $(\ell+1)$-Puiseux's package (see Equations (2.9)). We have

$$\sigma_{\mathcal{A},t} = \sum_{i=1}^{r} \lambda_{t,i} \frac{dx_i}{x_i} + \mu_t \frac{dz}{z} = \sum_{i=1}^{r} \lambda'_{t,i} \frac{d\tilde{x}_i}{\tilde{x}_i} + \mu'_t \, \phi^{-1} \tilde{z} \frac{d\tilde{z}}{\tilde{z}} \ . \tag{7.13}$$

Thus we can rewrite (7.11) as

$$\mathrm{crit}_{\mathcal{A}}(\omega) = \tilde{\boldsymbol{x}}^{\boldsymbol{r}} U \left\{ \sum_{i=1}^{r} P_i \frac{d\tilde{x}_i}{\tilde{x}_i} + \phi^{-1} \tilde{z} \, Q \frac{d\tilde{z}}{\tilde{z}} \right\} \ , \tag{7.14}$$

where $P_i, Q \in k[\tilde{z}]$ are given by

$$P_i = \sum_{t=0}^{M} \lambda'_{t,i} (\tilde{z} + \xi)^{M-t} \quad \text{and} \quad Q = \sum_{t=0}^{M} \mu'_t (\tilde{z} + \xi)^{M-t} \ . \tag{7.15}$$

Note that from (7.14) it follows that

$$\nu_{\tilde{\mathcal{B}}}(\mathrm{crit}_{\mathcal{A}}(\omega)) = \delta(\omega; \mathcal{A}; \gamma) \ .$$

By construction, for each index $1 \leq i \leq r$ we have

$$P_i = 0 \Longleftrightarrow \lambda'_{t,i} = 0 \text{ for } t = 0, 1, \ldots M \ .$$

In the same way we have

$$Q = 0 \Longleftrightarrow \mu'_t = 0 \text{ for } t = 0, 1, \ldots M \ .$$

Note that since $\sigma_{\mathcal{A},0} \neq 0$ we have $(P_1, P_2, \ldots, P_r, Q) \neq \mathbf{0}$. Consider the non-negative integer $\hbar$ defined by

$$\hbar := \min \{\mathrm{ord}(P_1), \mathrm{ord}(P_2), \ldots, \mathrm{ord}(P_r), \mathrm{ord}(Q) + 1\} \ . \tag{7.16}$$

Let us show that $\hbar \leq \chi(\omega; \mathcal{A}; \gamma)$. Suppose that $(P_1, P_2, \ldots, P_r) \neq \mathbf{0}$. We have

$$\min\{\mathrm{ord}(P_1), \mathrm{ord}(P_2), \ldots, \mathrm{ord}(P_r)\} \leq M = \left[\frac{\chi}{d}\right] \leq \chi \ ,$$

hence $\hbar \leq \chi$. Now, suppose $(P_1, P_2, \ldots, P_r) = \mathbf{0}$, so $Q \neq 0$. If $d \geq 2$ we have

$$\hbar = \mathrm{ord}(Q) + 1 \leq M + 1 = \left[\frac{\chi}{d}\right] + 1 \leq \left[\frac{\chi}{2}\right] + 1 \leq \chi \ .$$

On the other hand, if $d = 1$ (thus $M = \chi$) we have that $\mu'_M = 0$. Let us explain in detail this last affirmation. By assumption we have $\lambda'_{M,1} = \cdots = \lambda'_{M,r} = 0$. We also have that $\mu_M = 0$. In fact, we have

$$d \text{ divides } \chi \Rightarrow \mu_M = 0 \tag{7.17}$$

since $\sigma_{\mathcal{A},M}$ corresponds to the level ω_0 (and $f_0 = 0$). Looking at (7.12), (2.6) and (2.7) we obtain

$$\mathbf{0} = (\lambda'_{M,1}, \lambda'_{M,2}, \ldots, \lambda'_{M,r}) = (\lambda_{M,1}, \lambda_{M,2}, \ldots, \lambda_{M,r})\check{H}_\pi \ .$$

Since $\check{H}_\pi$ is an invertible matrix it follows that $\lambda_{M,1} = \cdots = \lambda_{M,r} = 0$. Looking again at (7.12) we conclude that $\mu'_M = 0$ as desired. In consequence $\mathrm{ord}(Q) \leq \chi - 1$ hence $\hbar \leq \chi$.

In view of the expression of $\mathrm{crit}_{\mathcal{A}}(\omega)$ given in (7.14) we have that all non-zero levels of $\mathrm{crit}_{\mathcal{A}}(\omega)$ in $\tilde{\mathcal{B}}$ are dominant with explicit value $\delta(\omega; \mathcal{A}; \gamma)$ and the lowest one is the one located at height $\hbar$.

The above arguments used to study the properties of $\mathrm{crit}_{\mathcal{A}}(\omega)$ in $\tilde{\mathcal{B}}$ also give that

$$\nu_{\tilde{\mathcal{B}}}(\omega - \mathrm{crit}_{\mathcal{A}}(\omega)) \geq \nu_{\tilde{\mathcal{B}}}(\mathrm{crit}_{\mathcal{A}}(\omega)) = \delta(\omega; \mathcal{A}; \omega) \ ,$$

and that all the levels of $\omega - \mathrm{crit}_{\mathcal{A}}(\omega)$ in $\tilde{\mathcal{B}}$ are not $\delta(\omega; \mathcal{A}; \omega)$-final dominant. Since $\omega = \mathrm{crit}_{\mathcal{A}}(\omega) + (\omega - \mathrm{crit}_{\mathcal{A}}(\omega))$ we conclude that

$$\nu_{\tilde{\mathcal{B}}}(\omega) = \delta(\omega; \mathcal{A}; \omega)$$

and that the level at height $\hbar$ of ω in $\tilde{\mathcal{B}}$ is $\delta(\omega; \mathcal{A}; \omega)$-final dominant.

Finally, after performing the γ-preparation $\tau : \tilde{\mathcal{B}} \to \mathcal{B}$ we obtain

$$\nu_{\mathcal{B}}(\omega) = \delta(\omega; \mathcal{A}; \omega) \ ,$$

and in the case that $\delta(\omega; \mathcal{B}; \omega) < \gamma$ we must have

$$\chi(\omega; \mathcal{B}; \omega) \leq \hbar \leq \chi(\omega; \mathcal{A}; \omega) \ .$$

□

7.4 Resonant conditions

Proposition 12 guarantees that the critical height cannot increase by means of a $(\ell + 1)$-Puiseux's package. Now we give conditions to assure that the critical height drops.

Let $\omega \in N_{\mathcal{A}}^{\ell+1}$ be a γ-prepared 1-form which is not pre-γ-final. As we did in the proof of Proposition 12 denote

$$\sigma_{\mathcal{A},t} := \sum_{i=1}^{r} \lambda_{t,i} \frac{dx_i}{x_i} + \mu_t \frac{dz}{z} \ , \quad (\boldsymbol{\lambda}, \mu) \in \mathbb{C}^{r+1} \setminus \{\mathbf{0}\} \ .$$

Now we establish the *resonant conditions*:

Resonant Condition (R1): We say that the condition (R1) is satisfied in $\mathcal{A}$ if

$$\delta(\omega; \mathcal{A}; \gamma) < \gamma \ , \quad \chi(\omega; \mathcal{A}; \gamma) = 1 \ , \quad d(\ell+1; \mathcal{A}) \geq 2 \ ,$$

and the following equivalent conditions are satisfied:

1. The coefficients of $\sigma_{\mathcal{A},0}$ satisfies

$$(\lambda_{0,1} : \lambda_{0,2} : \cdots : \lambda_{0,r} : \mu_0) = (p_1 : p_2 : \cdots : p_r : -d) \in \mathbb{P}_k^r \ ; \tag{7.18}$$

2. The 1-form $\mathrm{crit}_{\mathcal{A}}(\omega)$ can be written as

$$\mathrm{crit}_{\mathcal{A}}(\omega) = \mu_0 \, \boldsymbol{x}^{\boldsymbol{q}_1} z \frac{d\phi}{\phi} \ , \quad \mu_0 \in k^* \ . \tag{7.19}$$

Resonant Condition (R2): We say that the condition (R2) is satisfied in $\mathcal{A}$ if

$$\delta(\omega; \mathcal{A}; \gamma) < \gamma \ , \quad \chi(\omega; \mathcal{A}; \gamma) \geq 1 \ , \quad d(\ell+1; \mathcal{A}) = 1 \ ,$$

and the following equivalent conditions are satisfied:

1. For each index $1 \leq t \leq \chi$ the coefficients of $\sigma_{\mathcal{A},t}$ are

$$\begin{aligned} \lambda_{t,i} &= (-1)^t \xi^t \left[\binom{\chi}{t} \lambda_{0,i} + p_i \binom{\chi-1}{t-1} \mu_0 \right] \ , \quad t = 1, \ldots \chi; \\ \mu_t &= (-1)^t \binom{\chi-1}{t} \xi^t \mu_0 \, , \quad t = 1, \ldots \chi - 1 \ ; \end{aligned} \tag{7.20}$$

2. The 1-form $\mathrm{crit}_{\mathcal{A}}(\omega)$ can be written as

$$\mathrm{crit}_{\mathcal{A}}(\omega) = \boldsymbol{x}^{\boldsymbol{q}_\chi} (z - \xi \boldsymbol{x}^{\boldsymbol{p}})^\chi \left[\frac{d\boldsymbol{x}^{\boldsymbol{\lambda}_0}}{\boldsymbol{x}^{\boldsymbol{\lambda}_0}} + \mu_0 \frac{d\,(z - \xi \boldsymbol{x}^{\boldsymbol{p}})}{(z - \xi \boldsymbol{x}^{\boldsymbol{p}})} \right] \ . \tag{7.21}$$

Remark 23. In the definition of condition (R2) we have used the notation

$$\frac{d\boldsymbol{x}^{\boldsymbol{\lambda}}}{\boldsymbol{x}^{\boldsymbol{\lambda}}} := \sum_{i=1}^{r} \lambda_i \frac{dx_i}{x_i} \ .$$

This section is devoted to prove the following proposition:

Proposition 13. *Let $\omega \in N_{\mathcal{A}}^{\ell+1}$ a γ-prepared 1-form which is not pre-γ-final. Consider the $(\ell+1)$-nested transformation*

$$\mathcal{A} \xrightarrow{\pi} \tilde{\mathcal{B}} \xrightarrow{\tau} \mathcal{B}$$

where $\pi : \mathcal{A} \to \tilde{\mathcal{B}}$ is a $(\ell+1)$-Puiseux's package and $\tau : \tilde{\mathcal{B}} \to \mathcal{B}$ is a γ-preparation. Suppose that $\delta(\omega; \mathcal{B}, \gamma) < \gamma$. If in addition neither (R1) nor (R2) are satisfied in $\mathcal{A}$ then

$$\chi(\omega; \mathcal{B}, \gamma) < \chi(\omega; \mathcal{A}; \gamma) \ .$$

In the proof of this proposition we use some calculations made in the proof of Preposition 12. For short, denote by χ and χ' the critical heights $\chi(\omega; \mathcal{A}; \gamma)$ and $\chi(\omega; \mathcal{B}, \gamma)$ respectively, and denote by d the ramification index $d(\ell + 1; \mathcal{A})$.

The integer number $\hbar$ (defined in Equation (7.16)) is a bound for the new critical height χ'. It satisfies

$$\chi' \leq \hbar \leq \left[\frac{\chi}{d}\right] + 1 \ . \tag{7.22}$$

We study separately the cases $d = 1$ and $d \geq 2$.

The case $d \geq 2$. We have the following inequalities:

$$\begin{aligned} \left[\frac{\chi}{d}\right] + 1 \leq \frac{\chi}{d} + 1 \leq \frac{\chi}{2} + 1 < \chi \ , &\quad \text{if } \chi \geq 3 \text{ and } d \geq 2 \ ; \\ \left[\frac{\chi}{d}\right] + 1 = \left[\frac{2}{d}\right] + 1 = 1 \ , &\quad \text{if } \chi = 2 \text{ and } d > 2 \ . \end{aligned}$$

Therefore, except in the cases $\chi = 1$ or $\chi = d = 2$ the above inequalities and (7.22) give us $\chi' < \chi$.

Consider the case $\chi = d = 2$. We have $M = [\chi/d] = 1$. By (7.17) we have that $\mu_1 = 0$. Therefore

$$\begin{aligned} (P_1, P_2, \ldots, P_r, Q) &= (\tilde{z} + \xi) \left(\lambda'_{0,1}, \lambda'_{0,2}, \ldots, \lambda'_{0,r}, \mu'_0\right) + \left(\lambda'_{1,1}, \lambda'_{1,2}, \ldots, \lambda'_{1,r}, \mu'_1\right) \\ &= \phi \left(\lambda_{0,1}, \lambda_{0,2}, \ldots, \lambda_{0,r}, \mu_0\right) H + \left(\lambda_{1,1}, \lambda_{1,2}, \ldots, \lambda_{1,r}, 0\right) H. \end{aligned}$$

If some $P_i \neq 0$ we have that $\hbar \leq \mathrm{ord}(P_i) \leq 1$. Suppose $P_i = 0$ for $i = 1, \ldots, r$, thus $Q \neq 0$. We have that

$$(P_1, P_2, \ldots, P_r) = \mathbf{0} \Rightarrow \left(\lambda'_{1,1}, \lambda'_{1,2}, \ldots, \lambda'_{1,r}\right) = \mathbf{0} \ .$$

It follows from $\mu_1 = 0$ that

$$\left(\lambda'_{1,1}, \lambda'_{1,2}, \ldots, \lambda'_{1,r}\right) = \left(\lambda_{1,1}, \lambda_{1,2}, \ldots, \lambda_{1,r}\right) \check{H} \ .$$

Therefore we have

$$\left(\lambda'_{1,1}, \lambda'_{1,2}, \ldots, \lambda'_{1,r}\right) = \mathbf{0} \Rightarrow \left(\lambda_{1,1}, \lambda_{1,2}, \ldots, \lambda_{1,r}\right) = \mathbf{0} \ ,$$

since $\check{H}$ is invertible. Thus we have $\mu'_1 = 0$ which implies $\hbar = \mathrm{ord}(Q) + 1 = 1$.

Now assume $\chi = 1$. We have $M = 0$ so

$$\begin{aligned} (P_1, P_2, \ldots, P_r, Q) &= (\tilde{z} + \xi) \left(\lambda'_{0,1}, \lambda'_{0,2}, \ldots, \lambda'_{0,r}, \mu'_0\right) \\ &= \phi \left(\lambda_{0,1}, \lambda_{0,2}, \ldots, \lambda_{0,r}, \mu_0\right) H \ . \end{aligned}$$

If some $P_i \neq 0$ we have that $\hbar \leq \operatorname{ord}(P_i) \leq 0$. On the other hand we have

$$(P_1, P_2, \ldots, P_r) = \mathbf{0} \Leftrightarrow \left(\lambda'_{0,1}, \lambda'_{0,2}, \ldots, \lambda'_{0,r}\right) = \mathbf{0} \ .$$

By Equation (7.12) and (2.7) we have

$$\left(\lambda'_{0,1}, \lambda'_{0,2}, \ldots, \lambda'_{0,r}\right) = \left(\lambda_{0,1}, \lambda_{0,2}, \ldots, \lambda_{0,r}\right) \check{H} + \mu_0 \boldsymbol{\alpha}_0 \ ,$$

hence

$$\left(\lambda_{0,1}, \lambda_{0,2}, \ldots, \lambda_{0,r}\right) \check{H} + \mu_0 \boldsymbol{\alpha}_0 = \mathbf{0} \ .$$

Since $\check{H}$ is invertible, following Equation (2.8), we have that

$$\left(\lambda_{0,1}, \lambda_{0,2}, \ldots, \lambda_{0,r}, \mu_0\right) = -\frac{\mu_0}{d} \left(p_1, p_2, \ldots, p_r, -d\right) \ ,$$

so

$$\hbar = 1 \Leftrightarrow \text{ condition (R1) is satisfied } .$$

The case $d = 1$. First of all, recall that the matrix H of the $(\ell+1)$-Puiseux's package has the form

$$\left(\begin{array}{ccc|c} & & & 0 \\ & \check{H} & & \vdots \\ & & & 0 \\ \hline \check{p}_1 & \cdots & \check{p}_r & 1 \end{array}\right)$$

where $\check{\boldsymbol{p}} = \boldsymbol{p}\check{H}$ (see the end of Section 2.2.3). Note also that $M = \chi$. For all $0 \leq t \leq \chi$ denote

$$\left(\check{\lambda}_{t,1}, \check{\lambda}_{t,2}, \ldots, \check{\lambda}_{t,r}\right) := \left(\lambda_{t,1}, \lambda_{t,2}, \ldots, \lambda_{t,r}\right) \check{H} \ .$$

We have that

$$\begin{aligned}
(P_1, P_2, \ldots, P_r, Q) &= \sum_{t=0}^{\chi} \left(\lambda'_{t,1}, \lambda'_{t,2}, \ldots, \lambda'_{t,r}, \mu'_t\right) (\tilde{z} + \xi)^{\chi - t} \\
&= \sum_{t=0}^{\chi} \left(\lambda_{t,1}, \lambda_{t,2}, \ldots, \lambda_{t,r}, \mu_t\right) H \, (\tilde{z} + \xi)^{\chi - t} \\
&= \sum_{t=0}^{\chi} \left(\check{\lambda}_{t,1} + \check{p}_1 \mu_t, \check{\lambda}_{t,2} + \check{p}_2 \mu_t, \ldots, \check{\lambda}_{t,r} + \check{p}_r \mu_t, \mu_t\right) (\tilde{z} + \xi)^{\chi - t} \ .
\end{aligned}$$

On the other hand we have that $\hbar = \chi$ if and only if

$$\operatorname{ord}(P_i) \geq \chi \quad \text{for } i = 1, \ldots, r$$

and

$$\operatorname{ord}(Q) \geq \chi - 1 \ .$$

Since $\mu_\chi = 0$ (see Equation (7.17)), it follows that $\hbar = \chi$ if and only if

$$P_i = \left(\check{\lambda}_{0,i} + \check{p}_i \mu_0\right) \tilde{z}^{\chi} \quad \text{for } i = 1, \ldots, r$$

and

$$Q = (\tilde{z} + \xi) \, \mu_0 \tilde{z}^{\chi - 1} \ .$$

These last two equalities are equivalent to condition (R2) so, in the conditions of the proposition we have $\hbar < \chi$.

Remark 24. Proposition 13 give us necessary conditions for the critical height remains stable. Note that they are not sufficient conditions: in addition, it must happen that

$$\delta(\omega;\mathcal{B};\gamma)=\beta(\omega;\mathcal{B},\gamma)+\hbar\nu(z')\ ,$$

or, equivalently,

$$\nu_{\mathcal{B}}(\omega)=\nu_{\mathcal{B}}(\mathrm{crit}_{\mathcal{B}}(\omega))\ .$$

7.5 Reductions

As we did in Section 5.4 in the case of functions, we will complete the proof of Statement $T_3(\ell+1)$ by reductio ad absurdum.

Let $\mathcal{A}$ be a parameterized regular local model for K,ν, and let $\omega\in N_{\mathcal{A}}^{\ell+1}$ be a 1-form such that $\nu_{\mathcal{A}}(\omega\wedge d\omega)\geq 2\gamma$. We assume

1. ω is γ-prepared;
2. for any $(\ell+1)$-nested transformation $\mathcal{A}\to\mathcal{B}$ we have that ω is not pre-γ-final in $\mathcal{B}$.

The first assumption is possible thanks to Theorem 6. In this section we will see some implications of the second assumption and finally, in the next section, we will get a contradiction.

As we said, our main control invariant is the critical height $\chi(\omega;\mathcal{A};\gamma)$. By Proposition 13 we know that the critical height can only remain stable under a $(\ell+1)$-Puiseux's package if one of the resonant conditions is satisfied.

Lemma 13. *Suppose that condition (R1) is satisfied in $\mathcal{A}$. Consider a $(\ell+1)$-nested transformation*

$$\mathcal{A}=\mathcal{A}_0\xrightarrow{\pi_1}\tilde{\mathcal{A}}_1\xrightarrow{\tau_1}\mathcal{A}_1\xrightarrow{\pi_2}\cdots\xrightarrow{\pi_N}\tilde{\mathcal{A}}_N\xrightarrow{\tau_N}\mathcal{A}_N=\mathcal{B}$$

where each $\tau_i:\tilde{\mathcal{A}}_i\to\mathcal{A}_i$ is a γ-preparation and $\pi_j:\mathcal{A}_j\to\tilde{\mathcal{A}}_{j+1}$ is a $(\ell+1)$-Puiseux's package. If $\chi(\omega;\mathcal{B},\gamma)=\chi(\omega;\mathcal{A};\gamma)$ then condition (R2) is satisfied in $\mathcal{B}$.

Proof. Let $(\boldsymbol{x}_0,\boldsymbol{y}_0,z_0)$ be the coordinates in $\mathcal{A}_0$. Since (R1) is satisfied in $\mathcal{A}_0$ we have that

$$\mathrm{crit}_{\mathcal{A}_0}(\omega)=\mu\,\boldsymbol{x}_0^{\boldsymbol{q}_0}z_0\frac{d\phi_0}{\phi_0}\ ,$$

where $\phi_0=z_0^d/\boldsymbol{x}_0^{\boldsymbol{p}_0}$ is the $(l+1)$-th contact rational function in $\mathcal{A}_0$. Let $\xi_0\in k^*$ be the constant such that $\nu(\phi_0-\xi_0)>0$. After performing the $(l+1)$-Puiseux's package $\pi_1:\mathcal{A}_0\to\mathcal{A}_1$ we obtain

$$\mathrm{crit}_{\mathcal{A}_0}(\omega)=\mu\,\tilde{\boldsymbol{x}}_1^{\boldsymbol{r}}(\tilde{z}_1+\xi)^u\tilde{z}_1\frac{d\tilde{z}_1}{\tilde{z}_1}\ ,$$

where $\tilde{\boldsymbol{x}}_1$ and $\tilde{z}_1$ are the new variables and the exponents $\boldsymbol{r}$ and u are determined from Equations (2.3), and in particular we have $\nu(\tilde{\boldsymbol{x}}_1^{\boldsymbol{r}})=\delta(\omega;\mathcal{A}_0;\gamma)$. By assumption the critical height remains stable after the γ-preparation $\tau_1:\tilde{\mathcal{A}}_1\to\mathcal{A}_1$, so in $\mathcal{A}_1$ we have that

$$\mathrm{crit}_{\mathcal{A}_1}(\omega)=\boldsymbol{x}_1^{\boldsymbol{q}_1}z_1(\sigma_{\mathcal{A}_1,0}+\phi_1\sigma_{\mathcal{A}_1,1})\ ,$$

where $\boldsymbol{x}_1$ and $z_1 = \tilde{z}_1$ are the new variables, ϕ_1 is the $(l+1)$-th contact rational function in $\mathcal{A}_1$, the exponent $\boldsymbol{q}_1$ satisfies $\nu(\boldsymbol{x}_1^{\boldsymbol{q}_1}) = \delta(\omega; \mathcal{A}_0; \gamma)$ and, moreover, following Remark 14 we know that

$$\sigma_{\mathcal{A}_1,0} = \mu \frac{dz_1}{z_1} \ .$$

We see that condition (R1) is not satisfied in $\mathcal{A}_1$, so it must be satisfied condition (R2), thus

$$\sigma_{\mathcal{A}_1,1} = -\xi_1 \mu \frac{d\boldsymbol{x}^{\boldsymbol{p}_1}}{\boldsymbol{x}^{\boldsymbol{p}_1}} \ ,$$

where $z_1/\boldsymbol{x}^{\boldsymbol{p}_1} = \phi_1$ and $\nu(\phi_1 - \xi_1) > 0$. Equivalently, we have

$$\mathrm{crit}_{\mathcal{A}_1}(\omega) = \boldsymbol{x}_1^{\boldsymbol{q}_1} \mu \, d(z_1 - \xi_1 \boldsymbol{x}^{\boldsymbol{p}_1}) \ .$$

After performing the $(l+1)$-Puiseux's package $\pi_1 : \mathcal{A}_1 \to \tilde{\mathcal{A}}_2$ we obtain

$$\mathrm{crit}_{\mathcal{A}_1}(\omega) = \mu \, \tilde{\boldsymbol{x}}_2^{\boldsymbol{q}_1} \tilde{z}_2 \left(\frac{d\tilde{\boldsymbol{x}}_2^{\boldsymbol{p}_1}}{\tilde{\boldsymbol{x}}_2^{\boldsymbol{p}_1}} + \frac{d\tilde{z}_2}{\tilde{z}_2} \right) \ ,$$

where $\tilde{\boldsymbol{x}}_2 = \boldsymbol{x}_1$ and $\tilde{z}_2 = \phi_1 - \xi_1$ are the new variables. Again, since the critical height remains stable, after performing the γ-preparation $\tau_2 : \tilde{\mathcal{A}}_2 \to \mathcal{A}_2$ we have

$$\mathrm{crit}_{\mathcal{A}_2}(\omega) = \boldsymbol{x}_2^{\boldsymbol{q}_2} z_2 (\sigma_{\mathcal{A}_2,0} + \phi_2 \sigma_{\mathcal{A},2}) \ ,$$

where $\boldsymbol{x}_2$ and z_2 are the new variables, ϕ_2 is the $(l+1)$-th contact rational function in $\mathcal{A}_2$, the exponent $\boldsymbol{q}_2$ satisfies $\nu(\boldsymbol{x}_2^{\boldsymbol{q}_2}) = \delta(\omega; \mathcal{A}_1; \gamma)$ and, moreover, following Remark 14 we know that

$$\sigma_{\mathcal{A}_2,0} = \mu \left(\frac{d\boldsymbol{x}_2^{\boldsymbol{t}_2}}{\boldsymbol{x}_2^{\boldsymbol{t}_2}} + \frac{dz_1}{z_1} \right) \ ,$$

where $\boldsymbol{t}_2 = C_{\pi_2} \boldsymbol{p}_1$ being C_{π_2} the invertible matrix of non-negative integers related to π_2 (see Remark 14). Thus we have that $\boldsymbol{t}_2$ is a non-zero vector of non-negative integers, so

$$(t_{2,1} : t_{2,2} : \cdots : t_{2,r} : 1) \neq (p_{2,1} : p_{2,2} : \cdots : p_{2,r} : -d(\ell+1, \mathcal{A}_2)) \in \mathbb{P}_k^r \ ,$$

where $\boldsymbol{p}_2$ is given by $\phi_2 = z_2^{d(\ell+1,\mathcal{A}_2)}/\boldsymbol{x}_2^{\boldsymbol{p}_2}$, hence condition (R1) is not satisfied in $\mathcal{A}_2$ (note that all the integers in the left side term has the same sign while in the right side term there are negative and positive integers).

We have just check that in $\mathcal{A}_2$ condition (R2) must be satisfied. If we iterate the calculations made above, we obtain that in $\mathcal{A}_s$, for $2 \leq s \leq N$, the critical part is given by

$$\mathrm{crit}_{\mathcal{A}_s}(\omega) = \boldsymbol{x}_s^{\boldsymbol{q}_s} z_s (\sigma_{\mathcal{A}_s,0} + \phi_s \sigma_{\mathcal{A}_s,1}) \ ,$$

where $\boldsymbol{x}_s$ and z_s are coordinates in $\mathcal{A}_s$, ϕ_s is the $(l+1)$-th contact rational function in $\mathcal{A}_s$, the exponent $\boldsymbol{q}_s$ satisfies $\nu(\boldsymbol{x}_s^{\boldsymbol{q}_s}) = \delta(\omega; \mathcal{A}_{s-1}; \gamma)$ and

$$\sigma_{\mathcal{A}_s,0} = \mu \left(\frac{d\boldsymbol{x}_s^{\boldsymbol{t}_s}}{\boldsymbol{x}_s^{\boldsymbol{t}_s}} + \frac{dz_1}{z_1} \right) \ ,$$

where $\boldsymbol{t}_s = C_{\pi_s} \cdots C_{\pi_2} \boldsymbol{p}_1$ is a non-zero vector of non-negative integers. Thus we have that

$$(t_{s,1} : t_{s,2} : \cdots : t_{s,r} : 1) \neq (p_{s,1} : p_{s,2}, \cdots : p_{s,r} : -d(\ell+1, \mathcal{A}_s)) \in \mathbb{P}_k^r \ ,$$

hence condition (R1) is not satisfied in $\mathcal{A}_s$, which implies that condition (R2) is satisfied in $\mathcal{A}_s$ for all $1 \leq s \leq N$ as desired. $\square$

As we said, our main control invariant is the critical height $\chi(\omega; \mathcal{A}; \gamma)$. Proposition 12 allow us to make the following assumption:

> **Stability of the critical height.** Consider a $(\ell+1)$-nested transformations of the kind
>
> $$\mathcal{A} = \mathcal{A}_0 \xrightarrow{\pi_1} \tilde{\mathcal{A}}_1 \xrightarrow{\tau_1} \mathcal{A}_1 \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_N} \tilde{\mathcal{A}}_N \xrightarrow{\tau_N} \mathcal{A}_N = \mathcal{B}$$
>
> where each $\tau_i : \tilde{\mathcal{A}}_i \to \mathcal{A}_i$ is a γ-preparation and $\pi_j : \mathcal{A}_j \to \tilde{\mathcal{A}}_{j+1}$ is a $(\ell+1)$-Puiseux's package. We have
>
> $$\chi(\omega; \mathcal{B}, \gamma) = \chi(\omega; \mathcal{A}; \gamma) \ .$$

If there is such a transformation with $\chi(\omega; \mathcal{B}, \gamma) < \chi(\omega; \mathcal{A}; \gamma)$ we simply perform it.

Now, since the critical height $\chi(\omega; \mathcal{A}; \gamma)$ does not drop performing a $(\ell+1)$-Puiseux's package, we know that condition (R1) or (R2) are satisfied in $\mathcal{A}$.

In view of Lemma 13 we can make one more additional assumption

> **Stability of resonant condition (R2).** Consider a $(\ell+1)$-nested transformations of the kind
>
> $$\mathcal{A} = \mathcal{A}_0 \xrightarrow{\pi_1} \tilde{\mathcal{A}}_1 \xrightarrow{\tau_1} \mathcal{A}_1 \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_s} \tilde{\mathcal{A}}_N \xrightarrow{\tau_N} \mathcal{A}_N = \mathcal{B}$$
>
> where each $\tau_i : \tilde{\mathcal{A}}_i \to \mathcal{A}_i$ is a γ-preparation and $\pi_j : \mathcal{A}_j \to \tilde{\mathcal{A}}_{j+1}$ is a $(\ell+1)$-Puiseux's package. We have that condition (R2) is satisfied in $\mathcal{A}_j$ for every $j = 0, 1, \ldots, N$.

For a 1-form $\omega \in N_{\mathcal{A}}^{\ell+1}$, let us refer to the coefficient of dz by *the z-coefficient of ω in $\mathcal{A}$*. One of the features of condition (R2) is that $d(\ell+1; \mathcal{A}) = 1$. As a consequence we have the following key property:

> **Stability of the z-coefficient.** Consider a $(\ell+1)$-nested transformations of the kind
>
> $$\mathcal{A} = \mathcal{A}_0 \xrightarrow{\pi_1} \tilde{\mathcal{A}}_1 \xrightarrow{\tau_1} \mathcal{A}_1 \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_s} \tilde{\mathcal{A}}_N \xrightarrow{\tau_N} \mathcal{A}_N = \mathcal{B}$$
>
> where each $\tau_i : \tilde{\mathcal{A}}_i \to \mathcal{A}_i$ is a γ-preparation and $\pi_j : \mathcal{A}_j \to \tilde{\mathcal{A}}_{j+1}$ is a $(\ell+1)$-Puiseux's package. The z-coefficient of ω in $\mathcal{A}_j$ is the total transform of the z-coefficient of ω in $\mathcal{A}$.

Now, we will use Statement $T_4(\ell+1)$ in order to the z-coefficient of ω becomes γ-final (recall that $T_3(\ell) \Rightarrow T_4(\ell) \Rightarrow T_4(\ell+1)$). Note that a γ-preparation for

ω composed with a ℓ-nested transformation is still a γ-preparation for ω. Thus, we can determine a $(\ell+1)$-nested transformation of the kind

$$\mathcal{A} = \mathcal{A}_0 \xrightarrow{\pi_1} \tilde{\mathcal{A}}_1 \xrightarrow{\tau_1} \mathcal{A}_1 \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_s} \tilde{\mathcal{A}}_N \xrightarrow{\tau_N} \mathcal{A}_N = \mathcal{B}$$

where each $\tau_i : \tilde{\mathcal{A}}_i \to \mathcal{A}_i$ is a γ-preparation for both γ and f and $\pi_j : \mathcal{A}_j \to \tilde{\mathcal{A}}_{j+1}$ is either a $(\ell+1)$-Puiseux's package or an ordered change of the $(\ell+1)$-th coordinate, such that $f \in R_{\mathcal{B}}^{\ell+1}$ is γ-final. Proposition 11 guarantees that the critical height of ω can not increase. If $\chi(\omega; \mathcal{B}, \gamma) < \chi(\omega; \mathcal{A}; \gamma)$ we perform it an start again. If it remains stable we get a 1-form whose z-coefficient is γ-final.

Following Remark 24, we can also assume that

$$\nu_{\mathcal{A}}(\omega) = \nu_{\mathcal{A}}(\omega_\chi) \ .$$

Finally, just by performing a 0-nested transformation following Lemma 4 we can assume that the critical level has the form $\omega_\chi = \boldsymbol{x}^{\boldsymbol{q}} \tilde{\omega}_\chi$ where $\tilde{\omega}_\chi$ is log-elementary.

7.6 End of proof of Theorem 3

In this section we complete the proof of $T_3(\ell+1)$. In view of the considerations of the previous section we can assume that we have a parameterized regular local model $\mathcal{A}$, a value $\gamma \in \Gamma$ and a 1-form

$$\omega = \sum_{i=1}^{r} a_i \frac{dx_i}{x_i} + \sum_{j=1}^{\ell} b_j dy_j + zf\frac{dz}{z} \in N_{\mathcal{A}}^{\ell+1} \ , \quad \nu_{\mathcal{A}}(\omega \wedge d\omega) \geq 2\gamma \ ,$$

such that

1. $\delta(\omega; \mathcal{A}; \gamma) < \gamma$;
2. $\chi = \chi(\omega; \mathcal{A}; \gamma) > 0$;
3. $\nu_{\mathcal{A}}(\omega) = \nu_{\mathcal{A}}(\omega_\chi)$;
4. The critical level has the form $\omega_\chi = \boldsymbol{x}^{\boldsymbol{q}} \tilde{\omega}_\chi$ where $\tilde{\omega}_\chi$ is log-elementary;
5. The z-coefficient f is γ-final;
6. Condition (R2) is satisfied;
7. Properties 1, 2, 3, 4, 5 and 6 are stable for any $(\ell+1)$-nested transformation $\mathcal{A} \to \mathcal{B}$ such that ω is γ-prepared in $\mathcal{B}$.

We study separately different cases depending on the explicit value of the function f. In particular we will see that the previous assumptions implies $\chi = 1$.

7.6.1 The case $\nu_{\mathcal{A}}(f) \geq \nu_{\mathcal{A}}(\omega) + 2\nu(z)$.

Recall that

$$\nu_{\mathcal{A}}(\omega \wedge d\omega) \geq 2\gamma \Longrightarrow \nu_{\mathcal{A}}(\Delta_t) \geq 2\gamma$$

for all $t \geq 0$. In particular taking $t = 2\chi - 1$ we obtain

$$\nu_{\mathcal{A}}\left(\sum_{i+j=2\chi-1} \big(j\eta_j \wedge \eta_i + f_i d\eta_j + \eta_i \wedge df_j\big)\right) \geq 2\gamma \; . \tag{7.23}$$

Since $\nu_{\mathcal{A}}(f) \geq \nu_{\mathcal{A}}(\omega) + 2\nu(z)$ we have

$$\nu_{\mathcal{A}}(f_k) \geq \nu_{\mathcal{A}}(\omega) + 2\nu(z) \; , \quad \text{for all } k \geq 0 \; . \tag{7.24}$$

Since condition (R2) is satisfied we have

$$\nu_{\mathcal{A}}(\eta_{\chi-k}) \geq \nu_{\mathcal{A}}(\omega) + k\nu(z) \; , \quad \text{for all } k = 0, 1, \ldots, \chi \; . \tag{7.25}$$

Taking into account (7.24) and (7.25) we derive from (7.23) that

$$\nu_{\mathcal{A}}(\eta_{\chi-1} \wedge \eta_{\chi}) \geq 2\nu_{\mathcal{A}}(\omega) + 2\nu(z) \; .$$

Since $\eta_\chi = \boldsymbol{x}^{\boldsymbol{q}}\tilde{\eta}_\chi$, after factorizing $\boldsymbol{x}^{\boldsymbol{q}}$ in the above expression we obtain

$$\nu_{\mathcal{A}}(\eta_{\chi-1} \wedge \tilde{\eta}_{\chi}) \geq \nu_{\mathcal{A}}(\omega) + 2\nu(z) \; . \tag{7.26}$$

By Lemma 12 of truncated proportionality we know there is a function $g \in R^{\ell}_{\mathcal{A}}$ and a 1-form $\bar{\eta} \in N^{\ell}_{\mathcal{A}}$ with $\nu_{\mathcal{A}}(\bar{\eta}) \geq \nu_{\mathcal{A}}(\omega) + 2\nu(z)$ such that

$$\eta_{\chi-1} = g\,\tilde{\eta}_\chi + \bar{\eta} \; . \tag{7.27}$$

Note that (7.25) implies that $\nu_{\mathcal{A}}(g) \geq \nu_{\mathcal{A}}(\omega) + \nu(z)$. Let us write g as a power series

$$g = \sum_{(I,J)\in\mathbb{Z}^{r+\ell}_{\geq 0}} g_{IJ}\boldsymbol{x}^I\boldsymbol{y}^J \; , \quad f_{IJ} \in k \; .$$

Denote

$$g = G + H$$

where $G \in k[\boldsymbol{x}, \boldsymbol{y}] \subset R^{\ell}_{\mathcal{A}}$ is the polynomial

$$G = \sum_{\substack{(I,J)\in\mathbb{Z}^{r+\ell}_{\geq 0} \\ \nu(\boldsymbol{x}^I\boldsymbol{y}^J)\leq\nu_{\mathcal{A}}(\omega)+2\nu(z)}} g_{IJ}\boldsymbol{x}^I\boldsymbol{y}^J \; .$$

Now we perform the ordered change of coordinates $\mathcal{A} \to \tilde{\mathcal{A}}$ given by

$$\tilde{z} := z - \phi \; , \quad \phi := \frac{-1}{\chi}G \; .$$

As we saw in the proof of Proposition 11 we have that

$$\eta'_{\chi-1} = \eta_{\chi-1} + \chi\phi\eta_\chi + \phi^2(\cdots)$$

where $\eta'_{\chi-1} \in N^{\ell}_{\tilde{\mathcal{A}}}$ is the $(\chi-1)$-level of ω in $\tilde{\mathcal{A}}$. We have that

$$\eta'_{\chi-1} = g\,\tilde{\eta}_\chi + \bar{\eta} - G\eta_\chi + \phi^2(\cdots) = H\tilde{\eta}_\chi + \bar{\eta} + \phi^2(\cdots)\ .$$

Now, perform a γ-preparation $\tilde{\mathcal{A}} \to \mathcal{B}$. By definition of H we have that

$$\nu_{\mathcal{A}}(H) \geq \nu_{\mathcal{A}}(\omega) + 2\nu(z) \Longrightarrow \nu_{\mathcal{A}}(\eta'_{\chi-1}) \geq \nu_{\mathcal{A}}(\omega) + 2\nu(z)\ .$$

Since $\nu_{\mathcal{B}}(\eta_\chi) = \nu_{\mathcal{A}}(\eta_\chi) = \nu_{\mathcal{A}}(\omega)$ and condition (R2) must be satisfied in $\mathcal{B}$ we have that

$$\nu(z') \geq 2\nu(z)\ .$$

Iterating this procedure we obtain a sequence of parameterized regular local models whose $(\ell+1)$-th dependent variable has at least twice value that the previous one. The value of the $(\ell+1)$-th dependent variable can not be greater than

$$\frac{\gamma - \nu_{\mathcal{A}}(\omega)}{\chi}\ ,$$

since this implies that ω is pre-γ-final in such model. So, after finitely many steps we reach a model in which the value of the $(\ell+1)$-th dependent variable is greater than

$$\frac{\nu_{\mathcal{A}}(f) - \nu_{\mathcal{A}}(\omega)}{2}\ .$$

7.6.2 The case $\nu_{\mathcal{A}}(\omega) + \nu(z) \leq \nu_{\mathcal{A}}(f) < \nu_{\mathcal{A}}(\omega) + 2\nu(z)$.

Since f is dominant we have that

$$\nu_{\mathcal{A}}(f) \geq \nu_{\mathcal{A}}(\omega_1)\ .$$

On the other hand, since ω is γ-prepared we have

$$\nu_{\mathcal{A}}(\omega_1) = \nu_{\mathcal{A}}(\omega) + (\chi - 1)\nu(z)\ .$$

Since by assumption $\nu_{\mathcal{A}}(f) < \nu_{\mathcal{A}}(\omega) + 2\nu(z)$, we have that

$$\chi \leq 2\ .$$

Repeating the arguments of the previous case we obtain

$$\nu_{\mathcal{A}}(\eta_{\chi-1} \wedge \tilde{\eta}_\chi) \geq \nu_{\mathcal{A}}(f)\ .$$

Exactly as we did, we can perform an ordered change of coordinates followed by a γ-preparation and obtain a parameterized regular local model whose $(\ell+1)$-th dependent variable $\tilde{z}$ has value

$$\nu(\tilde{z}) > \nu_{\mathcal{A}}(f) - \nu_{\mathcal{A}}(\omega)\ .$$

7.6.3 The case $\nu_{\mathcal{A}}(\omega) \leq \nu_{\mathcal{A}}(f) < \nu_{\mathcal{A}}(\omega) + \nu(z)$.

In this case the only possibility is $\chi = 1$. Let $\epsilon > 0$ be the value given by

$$\epsilon := \nu_{\mathcal{A}}(\omega) - \nu_{\mathcal{A}}(f) \ .$$

Again, repeating the above arguments, we can perform an ordered change of coordinates followed by a γ-preparation and obtain a parameterized regular local model whose $(\ell + 1)$-th dependent variable $\tilde{z}$ has value

$$\nu(\tilde{z}) \geq \nu(z) + \epsilon \ .$$

Iterating, in finitely many steps we obtain a parameterized regular local model whose $(\ell + 1)$-th dependent variable z' has value

$$\nu(z') \geq \gamma - \nu_{\mathcal{A}}(\omega) \ ,$$

which implies that ω is pre-γ-final in such model in contradiction with our assumptions.

7.6.4 The case $\nu_{\mathcal{A}}(\omega) = \nu_{\mathcal{A}}(f)$.

We have just proved that $\chi = 1$ and $\nu_{\mathcal{A}}(\omega) = \nu_{\mathcal{A}}(f)$ are the only possibilities which are not in contradiction with our assumptions.

Since f is dominant, we can perform a 0-nested transformation given by Lemma 4 in order to obtain a parameterized regular local model in which f is a monomial in the independent variables times a unit. With one more application of Lemma 4 we can obtain a parameterized regular local model $\mathcal{A}'$ in which f divides ω. Denote $\gamma' = \gamma - \nu_{\mathcal{A}}(f)$. The 1-form $\omega' = f^{-1}\omega$ satisfies

$$\nu_{\mathcal{A}'}(\omega') = 0 \quad \text{and} \quad \nu_{\mathcal{A}'}(\omega' \wedge d\omega) \geq 2\gamma' \ .$$

So, replacing ω by ω', $\mathcal{A}$ by $\mathcal{A}'$ and γ by γ' we can "improve" our list of assumptions:

1. $\delta(\omega; \mathcal{A}; \gamma) < \gamma$;
2. $\chi = \chi(\omega; \mathcal{A}; \gamma) = 1$;
3. $\nu_{\mathcal{A}}(\omega) = \nu_{\mathcal{A}}(\omega_1) = 0$;
4. The critical level ω_1 is log-elementary;
5. The z-coefficient is $f = 1$;
6. Condition (R2) is satisfied;
7. Properties 1, 2, 3, 4, 5 and 6 are stable for any $(\ell+1)$-nested transformation $\mathcal{A} \to \mathcal{B}$ such that ω is γ-prepared in $\mathcal{B}$.

In this situation, we will show that it is always possible to determine an ordered change of the $(\ell + 1)$-th coordinate such that

$$\nu(z') \geq 2\nu(z) \ .$$

This is enough to get the desired contradiction, since iterating this procedure we necessarily reach a parameterized regular local model in which ω is pre-γ-final.

Since condition (R2) is satisfied we know that the critical part of ω can be written as

$$\mathrm{crit}_{\mathcal{A}}(\omega) = (z - \xi \boldsymbol{x}^{\boldsymbol{p}}) \left[\frac{d\boldsymbol{x}^{\boldsymbol{\lambda}}}{\boldsymbol{x}^{\boldsymbol{\lambda}}} + \frac{d\,(z - \xi \boldsymbol{x}^{\boldsymbol{p}})}{(z - \xi \boldsymbol{x}^{\boldsymbol{p}})} \right] ,$$

where $\boldsymbol{\lambda} \in k^r \setminus \{\boldsymbol{0}\}$, $\boldsymbol{p} \in \mathbb{Z}^r_{\geq 0} \setminus \{\boldsymbol{0}\}$, $\xi \in k^*$ and $\nu(z - \xi \boldsymbol{x}^{\boldsymbol{p}}) > \nu(z)$. This implies that

$$\eta_1 = \frac{d\boldsymbol{x}^{\boldsymbol{\lambda}}}{\boldsymbol{x}^{\boldsymbol{\lambda}}} + \bar{\eta}_1 ,$$

where $\bar{\eta}_1$ is not log-elementary, and

$$\eta_0 = -\xi \boldsymbol{x}^{\boldsymbol{p}} \left(\frac{d\boldsymbol{x}^{\boldsymbol{\lambda}}}{\boldsymbol{x}^{\boldsymbol{\lambda}}} + \frac{d\boldsymbol{x}^{\boldsymbol{p}}}{\boldsymbol{x}^{\boldsymbol{p}}} \right) + \bar{\eta}_0 , \quad \nu_{\mathcal{A}}(\bar{\eta}_0) \geq \nu(z) ,$$

where $\bar{\eta}_0$ is not $\nu(z)$-final dominant. Denoting

$$\sigma := \frac{d\boldsymbol{x}^{\boldsymbol{\lambda}}}{\boldsymbol{x}^{\boldsymbol{\lambda}}} \quad \text{and} \quad \psi_1 := \xi \boldsymbol{x}^{\boldsymbol{p}}$$

we have

$$\eta_1 = \sigma + \bar{\eta}_1$$

and

$$\eta_0 = -\psi_1 \sigma - d\psi_1 + \bar{\eta}_0 . \tag{7.28}$$

Let $\mathcal{A} \to \mathcal{A}_1$ be the ℓ-nested transformation given by Lemma 8. We have

$$\nu_{\mathcal{A}_1}(\bar{\eta}_1) > 0 \quad \text{and} \quad \nu_{\mathcal{A}_1}(\bar{\eta}_0) = \epsilon_1 > \nu(z) .$$

Consider the ordered change of the $(\ell+1)$-th coordinate $\mathcal{A}_1 \to \tilde{\mathcal{A}}_2$ given by

$$\tilde{z}_2 := z + \psi_1 .$$

In $\tilde{\mathcal{A}}_2$ the critical level is

$$\eta_1' = \sigma + \bar{\eta}_1 + \psi_1(\cdots)$$

and the 0-level is

$$\eta_0' = \bar{\eta}_0 + \psi_1^2(\cdots) .$$

If $\epsilon_1 \geq 2\nu(z)$ we are done. Indeed, if $\epsilon_1 \geq 2\nu(z)$ we have

$$\nu_{\tilde{\mathcal{A}}_2}(\eta_0') = \nu_{\tilde{\mathcal{A}}_2}(\bar{\eta}_0 + \psi_1^2(\cdots)) = \nu_{\mathcal{A}_1}(\bar{\eta}_0 + \psi_1^2(\cdots)) \geq 2\nu(z) ,$$

hence necessarily we have $\nu(\tilde{z}_2) \geq 2\nu(z)$, since after a γ-preparation condition (R2) must be satisfied.

Thus we have $\nu(z) < \epsilon_1 < 2\nu(z)$. As we said, after a γ-preparation $\tilde{\mathcal{A}}_2 \to \mathcal{A}_2$ condition (R2) must be satisfied, thus in $\mathcal{A}_2$ we have

$$\bar{\eta}_0 = -\psi_2\, \sigma - d\psi_2 + \bar{\bar{\eta}}_0 , \tag{7.29}$$

where

$$\psi_2 = \xi_2 \boldsymbol{x}_2^{\boldsymbol{p}_2} , \quad \xi_2 \in k^* \ , \nu(\boldsymbol{x}_2^{\boldsymbol{p}_2}) \geq \epsilon_1 > \nu(z) .$$

Since $\bar{\eta}_0$ is a element of $N^\ell_{\mathcal{A}_1} \subset N^\ell_{\mathcal{A}_2}$, we have that the equality given in (7.29) is also valid in $\mathcal{A}_1$, so we have that Equation (7.28) can be rewrite as

$$\eta_0 = -(\psi_1 + \psi_2)\sigma - d(\psi_1 + \psi_2) + \bar{\bar{\eta}}_0 \ . \tag{7.30}$$

We can iterate this method and obtain functions $\psi_3, \psi_4, \ldots \psi_k \in R^\ell_{\mathcal{A}_1}$ with increasing value. Since in $R^\ell_{\mathcal{A}_1}$ the amount of monomials with value lower than $2\nu(z)$ is finite, this procedure will provide an ordered change of the $(\ell+1)$-th coordinate $\mathcal{A}_1 \to \mathcal{A}'$ such that $\nu(z') \geq 2\nu(z)$.

Chapter 8

Proof of the main Theorem

In this chapter we end the proof of Theorem 1. Let us recall the precise statement. Let K be the function field of an algebraic variety defined over an algebraically closed field k of characteristic 0:

> **Theorem 1**: Let $\mathcal{F} \subset \Omega_{K/k}$ be a rational codimension one foliation of K/k. Given a rational archimedean valuation ν of K/k and a projective model M of K there exists a sequence of blow-ups with codimension two centers $\pi : \tilde{M} \longrightarrow M$ such that $\mathcal{F}$ is log-final at the center of ν in $\tilde{M}$.

As we detail in Chapter 2 this result is a consequence of Theorem 2:

> **Theorem 2**: Let $\mathcal{F} \subset \Omega_{K/k}$ be a rational codimension one foliation of K/k. Given a rational archimedean valuation ν of K/k and a projective model M of K there exists a nested transformation
>
> $$\mathcal{A} \longrightarrow \mathcal{B}$$
>
> such that $\mathcal{F}$ is $\mathcal{B}$-final.

Let $\mathcal{A} = \big(\mathcal{O}, (\boldsymbol{x}, \boldsymbol{y}, z)\big)$ be a parameterized regular local model for K, ν, where we denote the dependent variables as $(\boldsymbol{y}, z) = (y_1, y_2, \ldots, y_\ell, z)$, being $\ell = \operatorname{tr.deg}(K/k) - \operatorname{rat.rk}(\nu) - 1$. Take a generator of $\mathcal{F}_{\mathcal{A}}$

$$\omega = \sum_{i=1}^{r} a_i \frac{dx_i}{x_i} + \sum_{j=1}^{s-1} b_j dy_j + f dz \in \mathcal{F}_{\mathcal{A}} \subset \Omega_{\mathcal{O}/k}(\log \boldsymbol{x}) \ .$$

Since

$$\omega \in \Omega_{\mathcal{O}/k}(\log \boldsymbol{x}) \subset \Omega_{\mathcal{O}/k}(\log \boldsymbol{x}) \otimes_{\mathcal{O}} \hat{\mathcal{O}} = N_{\mathcal{A}}^{\ell+1} \ ,$$

we can apply Theorem 3 to ω. There are two possibilities:

1. For every $\gamma \in \Gamma$ there is a $(\ell+1)$-nested transformation

   $$\mathcal{A} \to \mathcal{B}_\gamma$$

   such that ω is γ-final recessive in $\mathcal{B}_\gamma$;

2. There is $\gamma \in \Gamma$ and a $(\ell+1)$-nested transformation

$$\mathcal{A} \to \mathcal{B}$$

such that ω is γ-final dominant in $\mathcal{B}$.

Suppose we are in the second case. Since ω is dominant in $\mathcal{B}$, we can perform a 0-nested transformation $\mathcal{B} \to \mathcal{C}$ given by Lemma 4 such that in $\mathcal{C}$ we have

$$\omega = Q\tilde{\omega}$$

where Q is a monomial in the independent variables and $\tilde{\omega}$ is log-elementary. Since the nested transformations are algebraic, we have that

$$\tilde{\omega} \in \mathcal{F}_{\mathcal{B}} \ ,$$

thus $\mathcal{F}$ is $\mathcal{B}$-final.

Now, suppose we are in the first case. We study separately the cases $f \neq 0$ and $f = 0$.

The case $f \neq 0$: Consider the decomposition of ω in z-levels:

$$\omega = \sum_{k=0}^{\infty} z^k \left(\eta_k + f_k \frac{dz}{z} \right) \ , \quad \eta_k \in N_{\mathcal{A}}^{\ell} \ , f_k \in R_{\mathcal{A}}^{\ell} \ .$$

By Theorem 4 we know that for each index $k \geq 1$ there are two possibilities:

1. For every $\gamma \in \Gamma$ there is a ℓ-nested transformation

$$\mathcal{A} \to \mathcal{B}_{\gamma}$$

such that f_k is γ-final recessive in $\mathcal{B}_{\gamma}$;

2. There is $\gamma \in \Gamma$ and a ℓ-nested transformation

$$\mathcal{A} \to \mathcal{B}$$

such that f_k is γ-final dominant in $\mathcal{B}$.

Suppose that for all $k \geq 0$ we are in the first case. In this situation, the same happens for the function f: fix a value γ, perform a ℓ-nested transformation such that all the functions f_k with $k \leq \gamma/\nu(z)$ are γ-final recessive and then perform a $(\ell+1)$-Puiseux's package. Since $f \in \mathcal{O} \subset R_{\mathcal{A}}^{\ell+1}$ it has a well defined value $\nu(f) \in \Gamma$ which keeps stable by means of birational morphisms, so f can not be γ-final recessive for any $\gamma \geq \nu(f)$ in any parameterized regular local model.

So let k_0 be the lowest index such that f_{k_0} can be transformed into a γ-final dominant function for some γ by means of a ℓ-nested transformation. Now take a value γ_1 such that

$$\gamma_1 > \gamma + k_0 \nu(z) \ .$$

Since $\omega \wedge d\omega = 0$ we have $\nu_{\mathcal{A}}(\omega \wedge d\omega) > 2\gamma$, so by Theorem 6 we know there is a ℓ-nested transformation $\mathcal{A} \to \mathcal{A}_1$ such that ω is γ_1-prepared in $\mathcal{A}_1$. Since

$$\nu_{\mathcal{A}_1}(f_{k_0}) \leq \gamma < \gamma_1 - k_0 \nu(z) \ ,$$

we have that

$$\delta(\omega;\mathcal{A}_1,\gamma_1)<\gamma_1\ ,$$

thus the critical height $\chi(\omega;\mathcal{A}_1,\gamma_1)$ is defined.

Now, perform a $(\ell+1)$-Puiseux's package $\mathcal{A}_1\to\tilde{\mathcal{A}}_2$. As we saw in the proof of Proposition 12, there is an integer $\hbar\le\chi(\omega;\mathcal{A}_1,\gamma_1)$ such that the $\hbar$-level of ω in $\tilde{\mathcal{A}}_2$ is dominant and it has the same explicit value than ω (which is exactly $\delta(\omega;\mathcal{A}_1,\gamma_1)$). Let $\tilde{z}_2$ be the $(\ell+1)$-th dependent variable in $\tilde{\mathcal{A}}_2$ and take a value γ_2 such that

$$\gamma_2>\delta(\omega;\mathcal{A}_1,\gamma_1)+\hbar\nu(\tilde{z}_2)\ .$$

Again, since $\omega\wedge d\omega=0$ we have $\nu_{\mathcal{A}}(\omega\wedge d\omega)>2\gamma_2$, so by Theorem 6 we know there is a ℓ-nested transformation $\tilde{\mathcal{A}}_2\to\mathcal{A}_2$ such that ω is γ_2-prepared in $\mathcal{A}_2$. Furthermore we know that

$$\delta(\omega;\mathcal{A}_2,\gamma_2)<\gamma_2$$

and

$$\chi(\omega;\mathcal{A}_2,\gamma_2)\le\chi(\omega;\mathcal{A}_1,\gamma_1)\ .$$

We can iterate this procedure as many times as we want and we obtain parameterized regular local models $\mathcal{A}_3,\mathcal{A}_4,\ldots$ such that

$$\chi(\omega;\mathcal{A}_t,\gamma_t)\le\chi(\omega;\mathcal{A}_{t-1},\gamma_{t-1})\ .$$

We know that these critical heights are always strictly greater than 0 since we are assuming that ω can not be transformed into a dominant 1-form.

As we saw in Section 7.6 the only possibility in this situation is that, after a finite number of steps, say T, we are in one of the cases treated in Subsection 7.6.3 or Subsection 7.6.4. Thus we have $\chi(\omega;\mathcal{A}_T,\gamma_T)=1$ and condition (R2) is satisfied in $\mathcal{A}_T$. After performing a 0-nested transformation $\mathcal{A}_T\to\mathcal{B}$ given by Lemma 4 if necessary, we have that

$$\omega=\tilde{\boldsymbol{x}}^{\boldsymbol{q}}\tilde{\omega}\ ,$$

with

$$\mathrm{crit}_{\mathcal{B}}(\tilde{\omega})=\boldsymbol{x}^{\boldsymbol{q}}\left(z-\xi\boldsymbol{x}^{\boldsymbol{p}}\right)\left[\frac{d\boldsymbol{x}^{\boldsymbol{\lambda}}}{\boldsymbol{x}^{\boldsymbol{\lambda}}}+\mu\,\frac{d\left(z-\xi\boldsymbol{x}^{\boldsymbol{p}}\right)}{\left(z-\xi\boldsymbol{x}^{\boldsymbol{p}}\right)}\right]\ ,$$

where $(\boldsymbol{\lambda},\mu)\in k^{r+1}\setminus\{\boldsymbol{0}\}$. If $\lambda\ne\boldsymbol{0}$ we have that $\tilde{\omega}$ is $\boldsymbol{x}$-log-canonical, hence $\mathcal{F}$ is $\mathcal{B}$-final. If $\lambda\ne 0$, as we saw in the proof of Lemma 13, after performing a $(\ell+1)$-Puiseux's package we reach the previous case ($\boldsymbol{\lambda}\ne\boldsymbol{0}$).

The case $f=0$: Thanks to the integrability condition, we have that this case corresponds to a foliation of lower dimensional type, it means, the foliation is an analytic cylinder over a foliation defined on a hypersurface. Let us check this assertion. Suppose without lost of generality that $a_1\ne 0$. Fix an index $2\le i\le r$. The coefficient of $\omega\wedge d\omega$ multiplying $\frac{dx_1}{x_1}\wedge\frac{dx_i}{x_i}\wedge dz$ is

$$a_i\frac{\partial a_1}{\partial z}-a_1\frac{\partial a_i}{\partial z}=-a_1^{-2}\frac{\partial a_i/a_1}{\partial z}$$

Due to the integrability condition it must be equal to zero, hence

$$\frac{\partial a_i/a_1}{\partial z}=0\ .$$

This is equivalent to say that there is a function $g_i \in k(\boldsymbol{x}, \boldsymbol{y})$ such that

$$a_i(\boldsymbol{x}, \boldsymbol{y}, z) = g_i(\boldsymbol{x}, \boldsymbol{y}) a_1(\boldsymbol{x}, \boldsymbol{y}, z) \ .$$

In the same way, fix an index $1 \leq j \leq \ell$. The coefficient of $\omega \wedge d\omega$ multiplying $\frac{dx_1}{x_1} \wedge dy_j \wedge dz$ is

$$b_j \frac{\partial a_1}{\partial z} - a_1 \frac{\partial b_j}{\partial z} = -a_1^{-2} \frac{\partial a_i / a_1}{\partial z} \ .$$

Again, it is equivalent to say that there is a function $h_j \in k(\boldsymbol{x}, \boldsymbol{y})$ such that

$$b_j(\boldsymbol{x}, \boldsymbol{y}, z) = h_j(\boldsymbol{x}, \boldsymbol{y}) a_1(\boldsymbol{x}, \boldsymbol{y}, z) \ .$$

Let $d(\boldsymbol{x}, \boldsymbol{y}) \in k[\boldsymbol{x}, \boldsymbol{y}]$ be the common denominator of $g_2, \ldots, g_r, h_1, \ldots, h_\ell$. Denote $G_i(\boldsymbol{x}, \boldsymbol{y}) = g_i(\boldsymbol{x}, \boldsymbol{y})/d(\boldsymbol{x}, \boldsymbol{y})$ and $H_j(\boldsymbol{x}, \boldsymbol{y}) = h_j(\boldsymbol{x}, \boldsymbol{y})/d(\boldsymbol{x}, \boldsymbol{y})$. We have that

$$\frac{a_1(\boldsymbol{x}, \boldsymbol{y}, z)}{d(\boldsymbol{x}, \boldsymbol{y})} \omega(\boldsymbol{x}, \boldsymbol{y}, z) = d(\boldsymbol{x}, \boldsymbol{y}) \frac{dx_1}{x_1} + \sum_{i=2}^{r} G_i(\boldsymbol{x}, \boldsymbol{y}) \frac{dx_i}{x_i} + \sum_{j=1}^{\ell} H_j(\boldsymbol{x}, \boldsymbol{y}) dy_j \ .$$

This 1-form belongs to $N_{\mathcal{A}}^{\ell}$ and it generates the foliation $\mathcal{F}$.

Iterating this method there are two possibilities

1. There is an index s, $1 \leq s \leq \ell$, and a 1-form $\omega' \in N_{\mathcal{A}}^{s}$, such that ω' is a generator $\mathcal{F}_{\mathcal{A}}$ and the coefficient of dy_s is not zero;
2. There is a 1-form $\omega' \in N_{\mathcal{A}}^{0}$, such that ω' is a generator $\mathcal{F}_{\mathcal{A}}$.

In both situations we have detailed how to obtain a parameterized regular local model $\mathcal{B}$ in which $\mathcal{F}$ is $\mathcal{B}$-final.

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